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EXTENDED MINIMUM-NUMBER THEOREMS OF CYCLIC AND SEXTACTIC POINTS ON A PLANE CONVEX OVAL

BY

S. MUKHOPADHAYA, 1929.

1. *Convex Arc and Convex Oval.*

The points of a convex arc S can be arranged in a certain order which we will call the *positive* order, with a first point A and a last point B, which are the *lower* and *upper* extremities of S. No point of S is repeated in the order and A is distinct from B.

The arc S is a *continuous* line and possesses the following properties :

If P(1), P(2),.........., P(n) be any n distinct points in order on S, they form the vertices in order of a convex n-gon.

If P(r) and P(s) be any two distinct points of S, P(s) being higher than P(r), then all the points of S which lie between P(r) and P(s) will lie on a given side (left side) of the directed line P(r)P(s), and all other points of S, excluding P(r) and P(s), will fall on the opposite side (right side) of the same directed straight line.

All points of the plane which lie on the left of AB or on AB, will constitute the *half-plane of S*.

A tangent exists at each point P of S and the positive direction of the tangent is such that each point of S, except P, lies on the right side of the tangent.

The positive direction of the tangent turns continuously on the right as P moves from A to B. The arc S is *rectifiable*.

The figure consisting of the convex arc S and its base AB will be called a *convex segment*.

A combination of two convex arcs, such that the lower and upper extremities of one are respectively the upper and lower extremities of the other, will be called a *double-convex*.

An *oval* is a double-convex having common tangents at the common extremities.

2. *Admissible Curves.*

A class X of curves C will be called *admissible* to S, if it possesses the following properties:

(*i*) Each of the curves C is uniquely determined by a fixed number n of arbitrary points on S. The number n will be called the *index* of X.

(*ii*) Each of the curves C, in the portion considered, is a convex arc in the half-plane of S, or such an arc, looked at from a projective point of view.

(*iii*) The curves C are *analytic*.

(*iv*) Two curves C and C' of class X with index n, do not intersect, in the portions considered in the half-plane of S, at more than $n-1$ points.

(*v*) The curves C of class X with index n vary in a continuous manner with the n determining points on S. A class X with index n may be denoted by X_n.

The variations of the n points on S determining a curve C will be always supposed to have been made in an *orderly* way, *i. e.*, with maintenance of their order on S. It may be observed that the points common to C and S will have the same order, including sense, on C and S. This is evident if the number of points common to C and S be not less than three, as C and S are both convex. More generally the condition of sameness of order including sense, on C and S, of the points common between them, will follow if we hypothetize that C and S between any two common points P and Q will fall on the same side of PQ.

It follows from condition of admissibility (v) that if by orderly variations of the n distinct points on S arbitrarily chosen, determining a curve C of class X with index n, a group of r ($\overline{>}$ n) points approached as their limit any fixed point P on S, *in any manner whatsoever*, the remaining $n-r$ points approaching at the same time other fixed limits, then C will approach a definite limiting form C_r. We will say that in this case C_r passes through r consecutive points of S at P.

Condition of admissibility (v) is equivalent to the supposition that S possesses *continuity of order n*.

3. *Orders of Incidence.*

Suppose C is an admissible curve of class X with index n, and suppose that r of the determining points on S have become consecutive at P. We will say in this case that C has *incidence of order* r at P on S.

When C passes through a *single* point P on S we will say that C has incidence of order *one* at P on S. When C does *not* pass through a point P of S, we will say that C has incidence of order *zero* at P on S.

At certain special points K of S it may happen that in every arbitrary double neighbourhood of K their exist $n+1$ intersections of a curve C of class X_n with S. Out of these $n+1$ points, only n can be arbitrarily chosen. The $n+1th$ point comes in *adventitiously* and will be called an *adventitious point*.

The curve C at such a point K will be said to have *adventitious incidence* on S at K of order $n+1$ and *strength* unity. Adventitious incidences of higher strength it will not be necessary to deal with. They will not affect the minimum-number theorems discussed in this paper and therefore may be ignored.

Where S has continuity of order n but *not higher*, an incidence of order $n+1$ can be distinguished from that of an order n by the fact that if in one case the curve C crosses S, in the other case it will not cross S, but an incidence of order higher than $n+1$ will not be distinguishable from one or other of the two abovementioned cases.

4. *Interstitial Incidence.*

We may make the following assumption:

If two curves C and C′ have incidence on S at P of orders r and r′, respectively, and if C′ fall between C and S, either in the immediate neighbourhood on the right of P or in that on the left of P, then $r' \geqq r$. This may be more succinctly stated that if C' has an incidence on S at P *interstitial* to C and S either on the left or on the right, then the order of incidence of C' on S at P cannot be less than that of C with S at P.

This assumption is based on geometrical considerations connected with the supposition of continuity of variations of C and C' along S and the *interlockability* of C and C' on S.

Two curves C and C' may be called *interlocked* on S if all the points common to C and C' in the half-plane of S are points of S.

Suppose C is determined by r arbitrary points P (1), P(2),..... P(r) and C' by r' arbitrary points P′(1), P′(2),...., P′(r'), on an arbitrary arc σ of S, of which P is the mid-point and which contains no further points of C or C'.

Suppose $P'(1)$ is the same as $P(1)$ and that C' is interstitial to C and S at $P(1)$. Suppose, if possible, $r' < r$. Then we can take $P'(2)$, $P'(3), \ldots, P'(r')$ identical with $P(2), P(3), \ldots, P(r')$, respectively. Evidently C' will also be interstitial to C and S at each of the points $P(2), P(3), \ldots, P(r')$, if we suppose C and C' are interlocked on σ. Consequently C' must either pass through $P(r'+1)$ or intersect S between $P(r')$ and $P(r'+1)$, which is against the hypothesis that C' has no further points on σ. Thus $r' \geqslant r$. By making σ vanish we arrive at our fundamental assumption.

The cases where either one of the points $P(1), P(2), \ldots, P(r)$ or one of the points $P'(1), P'(2), \ldots, P'(r')$ or one of the points of the former set as well as one of the points of the latter set happens to be adventitious, require separate consideration.

Suppose as before that the mid-point of σ is the point P where both sets of points become ultimately consecutive.

If an adventitious point occur only in the set $P(1), P(2), \ldots, P(r)$, the argument already given applies.

If an adventitious point occur only in the set $P'(1), P'(2), \ldots, P'(r')$, then if the supposition $r' < r$ were possible, we could identify $P(1)$, $P(2), \ldots, P(r')$ with $P'(1), P'(2), \ldots, P'(r')$ and the remaining $r-r'$ points $P(r'+1), \ldots, P(r)$ could be taken anywhere on σ above $P(r')$ in an arbitrary fashion. The argument already given will then apply.

If a point of the set $P(1), P(2), \ldots, P(r)$ be adventitious we can choose $P(1), P(2), \ldots P(r\text{-}1)$ arbitrarily on σ *below* P. The point $P(r)$ will then be adventitious and fall on S *above* P. Similarly if a point of the set $P'(1), P'(2), \ldots, P'(r')$ be adventitious we can choose $P'(1)$, $P'(2), \ldots P'(r'\text{-}1)$ arbitrarily on σ *below* P. The adventitious point $P'(r')$ will then fall on S *above* P. If we confine ourselves to the lower half of σ we can prove, by arguments already given, that $r'-1$ cannot be less than $r-1$. Whence it follows that $r' \geqslant r$.

5. *C-Ranges on S.*

If an admissible curve C of class X with index n, pass through $m(\geqslant n)$ points $P(1), P(2), \ldots, P(m)$, in order on S, and if there be no further intersections of C with S between $P(1)$ and $P(m)$ we have a *regular C-Range* on S of order m. The *strength* of a C-Range will be measured by $m-n$.

A regular C-Range of strength unity will be called a *prime* C-Range. A regular C-Range of strength greater than unity will be called a *composite* C-Range.

If C has incidence of order r with S at a point P of a regular C-Range, the point P will be considered as an r-ple point of the C-Range and as equivalent to r consecutive orderly intersections of C with S. A regular C-Range on S may be more briefly indicated by C/S.

At each *single* point P of C/S the curve C (measured in the positive sense) crosses S either from outside to inside of S or from inside to outside of S. In the former case the point P will be called an *in-point* and in the latter case an *out-point* of C/S. An in-point may also be called a *positive element* and an out-point a *negative element* of C/S. It should be borne in mind that the order of the points C/S, as well as the *sense* of the order, is the same on C and S.

The successive elements of C/S, when they are all distinct, are alternately of contrary signs.

An r-ple point P^r of C/S is to be looked upon as the juxtaposition of r consecutive points of alternately contrary signs. If r be odd the first and last points of P^r are of *similar* signs and when r is *even* they are of *contrary* signs.

If at a multiple point P^r, the curve C approaches S from *above* S, the first point of P^r is positive and if C approaches S from *below* S, it is negative. If at P^r the curve C when leaving S goes *below* S the last point of P^r is positive, and if it goes *above* S the last point of P^r is negative.

Two C-Ranges on S will be called *similar* if the first and last elements of one are similar in sign to the first and last elements, respectively, of the other. Two C-Ranges on S will be called *contrary* if the first and last elements of one are contrary in sign to the first and last elements, respectively, of the other.

Two C-Ranges will be called *infra-similar* or *infra-contrary* according as the first elements of the two are similar or contrary. Two C-Ranges on S will be called *supra-similar* or *supra-contrary* according as the last elements of the two are similar or contrary. A C-Range will be called positive or negative according as its first element is positive or negative.

If P and Q be any two points on C or S, Q being above P, the part of C or S which lies between P and Q, both inclusive, will be denoted by C(P Q) or S(P Q). If P and Q be denoted by P(r) and P(s), C(P Q) and S(P Q) may be denoted more briefly by $C(r, s)$ and $S(r, s)$. Any particular point on $S(r, s)$ may be denoted by P(r, s) or P(r s), omitting the punctuation. Any particular point on S between

$P(r\ s)$, and $P(s\ t)$ may be denoted by $P(r\ s\ t)$. Similarly any particular point on S between $P(r\ s\ t)$ an $P(s\ t\ u)$ may be denoted by $P(r\ s\ t\ u)$, and so on.

A convex segment consisting of $C(P\ Q)$ and $P\ Q$ will be denoted by $C(\overline{PQ})$ and one consisting of $C(r,\ s)$ and $P(r)\ P(s)$ will be denoted by $C(r,\ s)$.

A prime range on S pertaining to C will be denoted by $\{C/S\}$.

6. *Intersections of two Convex Segments.*

Suppose $C(P\ Q)$ is a convex arc and $C(\overline{PQ})$ the corresponding convex segment, where by our conventions $C(P\ Q)$ lies on the left side of $P\ Q$. Suppose $C'(\overline{P'Q'})$ is another convex segment in the half-plane of $C(P\ Q)$.

It follows from the elementary properties of a convex arc that if one of the points P', Q' lie inside $C(\overline{PQ})$ and the other outside in its half-plane, the line $P'\ Q'$ will cut $C\ (P\ Q)$ at a single point R and conversely.

It also follows that $C(\overline{PQ})$ and $C'(\overline{P'Q'})$ being closed figures intersect one another at an even number of points, supposing that they have only a finite number of points common. Hence since $P'\ Q'$ and $C\ (P\ Q)$ intersect at only one point and $C'\ (P'\ Q')$ does not meet $P\ Q$ at all, the convex arcs $C\ (P\ Q)$ and $C'\ (P'\ Q')$ intersect at an odd-number of points.

If O be a common point of $C\ (P\ Q)$ and $C'\ (P'\ Q')$ and P' lie inside $C\ (\overline{PQ})$ and Q' out-side, then O will lie on $C\ (P\ Q)$ below R, as O is on the left of $P'\ Q'$. If P' be outside and Q' inside then O will lie on $C\ (P\ Q)$ above R.

Conversely, if $C'\ (P'\ Q')$ lie in the half-plane of $C\ (P\ Q)$ and intersect $C\ (P\ Q)$ at a single point, or at an odd number of points, then P' and Q' lie on opposite sides of $C\ (P\ Q)$.

THEOREM I.

If $P(1)$, $P(2)$,....,$P(n+1)$ be the elements of a prime range $C/S\ (1,\ n+1)$ where C is an admissible curve of class X_n, and C is another curve of the same class X_n, determined by any n arbitrary points $P\ (1,\ 2)$, $P(2\ 3)$,....,$P(n,\ n+1)$ on S, then the regular range $C'/S(1,\ n+1)$ is similar to $C/S\ (1,\ n+1)$.

First suppose all the elements of $C/S\ (1,\ n+1)$ are distinct. Also suppose that none of the n arbitrary points, which interpolate $C/S\ (1,\ n+1)$ overlap any of the elements of $C/S\ (1,\ n+1)$.

As P (1, 2),....,P(n, $n+1$) are in order on S, they are also in order on C'.

P (r, $r+1$) and P ($r+1$, $r+2$), $r=1, 2, \ldots, n-1$, are on opposite sides of C, in the half-plane of C (r, $r+2$), as S crosses C only once between these points. Therefore the part of C' which passes between P (r, $r+1$) and P ($r+1$, $r+2$) will intersect C (r, $r+2$), in at least one point O_r.

The $n-1$ points O_1, $\mathrm{O}_2, \ldots \mathrm{O}_{n-1}$ thus obtained on the $n-1$ successive parts of C' between P(1, 2) and P(n, $n+1$) are necessarily in order on C'. They are therefore also in order on C, between P(1) and P($n+1$).

But C and C' cannot intersect at more than $n-1$ points in the half-plane of $S(1, n+1)$,

Therefore O_1, $\mathrm{O}_2, \ldots, \mathrm{O}_{n-1}$ are all the intersections of C and C' in the half-plane of S (1, $n+1$), in other words, C and C' cannot again intersect in the half-plane of S (1, $n+1$).

Suppose all the elements of C'/S (1, $n+1$) are P′(1), P′(2)...P′(m) These include the n points P (1, 2), P(2,3),....P(n, $n+1$). Consequently m is not less than n. P′(1) cannot coincide with P(1) for then C' would intersect C again at P(1), which is below P(1, 2) on C' and therefore much more below the lowest intersection O_1 of C' with C.

Again P(1) and P′(1) are similar elements, for if P (1) and P′ (1) were contrary, then C' would intersect C again in the half-plane of S (1, $n+1$) at some point O which is below P(1, 2) on C' and therefore much more below O_1 the lowest intersection of C' with C.

In the same way it is shewn that P ($n+1$) and P′(m) cannot coincide and are similar.

Thus it is completely proved that C/S (1, $n+1$) and C'/S (1, $n+1$) are similar. The order of C/S (1, $n+1$) is $n+1$ and the order of $C'/S(1, n+1)$ is not less than n. But two similar ranges have either the same order or differ in order by an even number. Hence the order of C'/S (1, $n+1$) is either $n+1$ or $n+1+2p$, where p is an integer. Thus either $C'/S(1, n+1)$ is a prime range or a composite range of strength $1+2p$.

The cases where some of the elements of C/S (1, $n+1$) are multiple or where some of the arbitrarily chosen points P(1, 2)...,P(n, $n+1$) coincide with some of these elements can be individually treated on similar lines and present no inherent difficulties. These cases may also be considered as orderly limits of the case treated above and deduced directly from it.

7. *Line of Mean Succession of Prime Ranges.*

If in Theorem I we take $P(1,2)$, $P(2,3),\ldots,P(n,\ n+1)$ to be the mid-points of $S(1,2)$, $S(2,3),\ldots,S(n,\ n+1)$, respectively, the curve C' of the same class and index as C determined by these mid-points will be called the *mean associate* of $C/S\ (1,\ n+1)$.

The range $C'/S\ (1,n+1)$ which will be similar to $C/S\ (1,n+1)$ will be called the *mean derived* of the latter.

The mean derived of a given prime range $\{P\}$ may be itself prime. In which case it may be denoted by $\{P'\}$ and called the *mean successor of* $\{P\}$.

If the mean derived of $\{P\}$ be composite and of strength $1+2p$ it will consist of $n+1+2p$ elements $P'(1)$, $P'(2),\ldots,P'(n+1+2p)$. These can be grouped into $2p+1$ successive prime ranges.

$$\begin{array}{l} P'(1),\ \ldots,\ P'(n+1), \\ P'(2),\ \ldots,\ P'(n+2), \\ \ldots\quad \ldots\quad \ldots \\ P'(1+2p),\ldots,P'(n+1+2p), \end{array}$$

which will be alternately *similar* and *contrary* to $\{P\}$. The first and last of the group which may be called the *header* and *tailer*, will be always similar to $\{P\}$. Any one of these $1+2p$ prime ranges, will ebcalled a *mean successor* of $\{P\}$.

If $\{P\}$, $\{P'\}$, $\{P''\},\ldots,\{P^{(n)}\},\ldots$be a sequence of prime ranges such that $\{P'\}$ is a mean successor of $\{P\}$ and $\{P^{(r+1)}\}$ of $\{P^{(r)}\}$, $r=1,2,3,\ldots$then this sequence will be called a *line of mean succession* of prime ranges.

If $\{P\}$, $\{P'\}$, $\{P''\},\ldots$be all similar, the line of mean succession which they form will be called *pure*. Otherwise the line will be called *mixed*, specially if the successors of no successor are all similar.

8. *X-points, positive and negative.*

A point of K of S will be called an X-point of given sign and index n, if in every arbitrary neighbourhood σ, extending to both sides of K, there exists a prime range C/σ of the given sign pertaining to a curve C of class X with index n.

An X-point is either positive or negative. An X-point which is both positive and negative will be supposed not to exist. The existence of such an X-point would imply the existence of an infinity of X-points of either signs in the immediate vicinity and no minimum theorem would require proof.

THEOREM II.

Every pure line of mean succession of prime ranges on S, derived from a given prime range {P} of given sign, pertaining to an admissible curve C of class X with index n, lends to a definite X-point of the given sign and index n.

Let {P}, {P′}, {P″}.......denote a pure line of mean succession of prime ranges derived from {P} and s, s', s'',....... their respective *laps* on S, that is, the arcual distances along S from lowest to highest element of the respective ranges.

It follows from (7) that s' falls within s, s'' within s', and so on. It can be shewn that s, s', s''.......form a sequence of zero limit. A proof of this for the case $n=5$ has been given elsewhere (see *Generalized Form of Böhmer's Theorem, Collected Geometrical Papers, Part I*, pp. 42-45). The method of proof given there is general and can be extended to any value of n.

There is only one point of S common to s, s', s'',.... by *Bolzano's Principle*. This unique point is an X-point of index n and given sign.

Cor. To every prime range of a given sign pertaining to a curve C of class X_n, corresponds at least one X_n-point of the given sign lying in the lap of the range.

9. *Bi-incidence on S.*

Let C^r_{n-r}, $r=0, 1, 2,\dots, n$, denote an admissible curve of class X_n, having incidence on S at A and B of orders $n-r$ and r, respectively. Such a curve will be called a curve *bi-incident* on S, the order of the bi-incidence being n. The sum of the orders of the incidences of each of these curves at A and B is n, the order of the bi-incidence.

The curves of class X_n which have bi-incidence on S(AB) of order n, can be serially written as:

$$C^0_n,\ C^1_{n-1},\dots\dots,C^r_{n-r},\dots C^n_0,$$

Two consecutive curves C^r_{n-r}, C^{r+1}_{n-r-1} have $n-r-1$ consecutive points common at A and r consecutive points common at B. They have thus altogether $n-1$ points common, and consequently cannot have any point common *between* A and B.

A curve C^r_{n-r} will be said to be *clearly* bi-incident on S (AB) if it has no point common with S *between* A and B.

A curve C^r_{n-r} will be called *adventitiously* bi-incident on S(AB), if it has one or more points common with S *between* A and B.

Two consecutive curves C^r_{n-r} and C^{r+1}_{n-r-1} can be looked upon as interlocked on S as all the points common between them are points of S.

THEOREM III.

Two consecutive curves C^r_{n-r} and C^{r+1}_{n-r-1} which have clear bi-incidence on $S(AB)$ will fall on different sides of S between A and B.

Suppose C^r_{n-r} falls *above* S between A and B. Then C^{r+1}_{n-r-1} will fall *below* S.

C^{r+1}_{n-r-1} cannot fall above C^r_{n-r} at B as then C^r_{n-r} would be *interstitial* to C^{r+1}_{n-r-1} and S at B and have at the same time a lower order of incidence, namely, r, on S at B than C^{r+1}_{n-r-1} has. C^{r+1}_{n-r-1} cannot be interstitial to C^r_{n-r} and S at B as then it would be also interstitial to C^r_{n-r} and S at A and have at the same time an order of incidence on S at A less than the order of incidence of C^r_{n-r} on S at A.

Consequently C_{n-r-1}^{n+1} will fall below S at B, and therefore also below S at A as the bi-incidence is *clear*.

Cor. (i). *If every one of the series $C^0_n, \ldots C^r_{n-r}, \ldots C^n_0$ have clear bi-incidence on S (A B), they will fall alternately on different sides of S.*

Cor. (ii). *The curves C^0_n and C^n_0 in the above corollary will fall on the same side or different sides of S according as n is even or odd.*

Cor. (iii). *The ranges C^r_{n-r}/S, $r=0,1,2,\ldots n$, are all similar, if the bi-incidences be all clear.*

This is evident, since if C^r_{n-r} fall above S at A, C^{r+1}_{n-r-1} will fall below S at A, that is, if the $n-r$th element of C^r_{n-r}/S be *negative* the $n-r-1$th element of C^{n-r-1}_{r+1}/S will be *positive* and the two ranges have the same order.

Cor. (iv). *Theorem III also evidently holds for two curves C^r_{n-r+x} and C^{r+1+y}_{n-r-1} of class X_n, the former possessing adventitious incidence of strength x at* A *and the latter possessing adventitious incidence of strength y at* B.

If C^r_{n-r} possess adventitious incidence at B or if C^{r+1}_{n-r-1} possess adventitious incidence at A, the two curves will be identical as they will possess altogether more than $n-1$ points common. Such possibilities will not affect the minimum-number theorems.

THEOREM IV.

If C^0_n and C^n_0 be two admissible curves of class X_n, which have no point common with S between A and B, and if they fall on different sides of S when n is even or if they fall on the same side of S when n is odd, the ranges C^0_n/S and C^n_0/S are contrary and conversely.

This theorem follows directly from definitions of positive and negative elements and those of similar and contrary ranges and does not stand in need of a formal proof.

THEOREM V.

If C^0_n/S (AB) and $C^n_0/S(AB)$ be two contrary ranges of order n, pertaining to two admissible curves C^0_n and C^n_0 of class X_n, then there exists a first curve of the series C^r_{n-r}, $r=1, 2, \ldots, n-1$, which is adventitiously bi-incident on S(AB).

Since $C^0_n/S(\mathrm{AB})$ and C^n_0/S (AB) are, by hypothesis, each of order n, they are *clearly* bi-incident on S (AB).

If all the curves of the series C^n_{n-r}, $r=0, 1, 2, \ldots, n$, were *clearly* bi-incident on S (AB), the ranges C^r_{n-r}/S would be all similar.

But the ranges C^0_n/S and C^n_0/S are, by hypothesis, contrary. Hence there exists a first curve of the series C^r_{n-r}, $r=1, 2, \ldots, n-1$, which is adventitiously bi-incident on S(AB).

Cor. If the first curve of the series C^r_{n-r}, $r=1, 2, \ldots, n-1$, which is adventitiously bi-incident on S be C^k_{n-k}, then the range C^k_{n-k}/S is infra-similar to C^0_n/S.

The ranges C^r_{n-r}/S, $r=0, 1, 2, \ldots k-1$, are all similar, as they are a succession of clear ranges.

Let the first point of C^k_{n-k} which falls between A and B on S be L. Let s denote S (AL).

The two curves C^{k-1}_{n-k+1} and C^k_{n-k} have $k-1$ consecutive points common with S at B.

Consider the series of curves of class X_{n-k+1} which have bi-incidence on s of order $n-k+1$ and pass through $k-1$ consecutive points at B, which are outside the half-plane of s.

Let C^{k-1}_{n-k+1} and C^{k}_{n-k} as members of this series be denoted by c^{0}_{n-k+1} and c^{1}_{n-k}. They are two consecutive curves of the series which are both *clearly* bi-incident on s.

Consequently c^{0}_{n-k+1}/s and c^{1}_{n-k}/s are similar ranges of order $n-k+1$. In other words C^{k-1}_{n-k+1}/S and C^{k}_{n-k}/S are infra-similar But C^{k-1}_{n-k+1}/S is similar to C^{0}_{n}/S. Therefore C^{k}_{n-k}/S and C^{0}_{n}/S are infra-similar.

THEOREM VI.

If C^{0}_{n}/S *and* C^{n}_{0}/S *be two contrary ranges of order n, where n is the index of the class to which* C^{0}_{n} *and* C^{n}_{0} *belong, then there exists on S a prime range belonging to a curve of the series* C^{r}_{n-r}, $r=1, 2, \ldots, n-1$, *which is infra-similar to* C^{0}_{n}/S *and supra-similar to* C^{n}_{0}/S.

If C^{k}_{n-k} be the first curve of the series to have adventitious bi-incidence on S, then we have shown that C^{k}_{n-k}/S is infra-similar to C^{0}_{n}/S. *Cor. Theorem V.*

But C^{k}_{n-k}/S is either a prime range or a composite range, as it has more than n elements.

In the first case we have a prime range on S infra-similar to C^{0}_{n}/S and therefore supra-similar to C^{n}_{0}/S, as C^{0}_{n}/S and C^{n}_{0}/S are contrary ranges of the same order n.

In the second case the header of the composite range will be a prime range infra-similar to C^{0}_{n}/S and therefore supra-similar to C^{n}_{0}/S.

Cor. If C^{0}_{n}/S (PQ) *and* C^{n}_{0}/S *(PQ) be two contrary ranges of order n pertaining to two curves* C^{0}_{n}/S(PQ) *and* C^{n}_{0}/S (PQ) *of class* X_n, *clearly bi-incident on S* (PQ), *then there exists on S between* P *and* Q *an* X_n*-point, infra-similar to* C^{0}_{n}/S (PQ) *and supra-similar to* C^{n}_{0}/S (PQ).

THEOREM VII.

If $C/S \equiv P(1), P(2), \ldots, P(n)$, $(n>3)$, *be a regular range on S of order n, pertaining to a circle C, then there exists on S, between P(1) and P(n), at least n−3 cyclic points, of alternately contrary signs.*

We will suppose that the circle determined by any three points on S varies in a continuous manner with the three determining points, in other words, that S has continuity of order 3.

The class X_3 of circles determined by any three arbitrary points on S is obviously an admissible one.

If we fix two points O_1 and O_2 on C, not in the half-plane of $S(1, n)$, then the class X_1 of circles which pass through the two fixed points O_1 and O_2 and any arbitrary point of S is obviously also an admissible one.

Again if we fix one point O_1 on C, not in the half-plane of $S(1, n)$, then the class X_2 of circles which pass through the fixed point O_1 and any two arbitrary points of S is obviously also an admissible one.

Consider first the class X_1 of circles. The circle C has bi-incidence on $S(1, 2)$. Suppose C cuts S positively at P(1) and therefore negatively at P(2). Hence there exists a positive X_1-point P(1, 2) on S between P(1) and P(2). Similarly there exists a negative X_1-point P(2, 3) between P(2) and P(3). Thus there exist $n-1$ X_1-points alternately positive and negative between P(1) and P(n).

Next consider the class X_2 of circles. The circles $C^0{}_2$ and $C^2{}_0$ of this class have bi-incidence on $S(12, 23)$, and the ranges pertaining to $C^0{}_2$ and $C^2{}_0$ on this arc are of contrary signs, that belonging to the former being positive. Hence there exists a positive X_2-point P(1, 2, 3) on S between P(1,2) and P(2, 3). Similarly there exists a negative X_2-point P(2,3,4) on S between P(2,3) and P(3,4).

Consider lastly the class X_3 of circles. The circles $C^0{}_3$ and $C^3{}_0$ of this class have bi-incidence on $S(123, 234)$ and the ranges pertaining to $C^0{}_3$ and $C^3{}_0$ on this arc have contrary signs, that belonging to the former being positive. Hence there exists a positive X_3-point P(1,2,3,4) on S between P(1,2,3) and P(2,3,4). Similarly there exists a negative X_3-point P(2,3,4,5) on S between P(2,3,4) and P(3,4,5). In this way there exist on S, between P(1) and P(n), $(n-3)$ X_3-points alternately positive and negative, which proves the theorem.

THEOREM VIII.

If $C/S \equiv P(1)$, $P(2)$........ $P(n)$, $(n > 5)$, be a regular range on S of order n, *pertaining to a conic C, then there exists on S, between $P(1)$ and $P(n)$, at least $n-5$ sextactic points, of alternately contrary signs.*

We will suppose that the conic determined by any five points on S varies in a continuous manner with the five determining points; in other words, that S has continuity of order 5.

If we fix r points $O_1, O_2, \ldots\ldots, O_r$, $r=1,2,3,4,0$, on C not in the half-plane of $S(1,n)$, then the class X_{5-r} of conics which pass through the r fixed points on C and $5-r$ arbitrary points on S is obviously an admissible one. We can prove successively on lines similar to that adopted in Theorem VII, that if C cut S positively at P(1), then there exist on S at least $n-1$ X_1-points, $n-2$ X_2-points, $n-3$ X_3-points, $n-4$ X_4-points and finally $n-5$ X_5-points, alternately positive and negative. Thus the theorem is proved.

THEOREM IX.

If a circle C intersect an oval V at $2n$ points ($n \geq 2$), then there exist at least $2n$ cyclic points, in order on V, of alternately contrary signs, provided the oval has continuity of order 3.

To fix ideas suppose $n=3$ and that the intersections taken in order are P(1), P(2)……P(6), the point P(1) being a positive intersection.

Let X_3 denote the class of circles through any three arbitrary points of V. Let $X_1(4,5)$ denote the class of circles through the two fixed points P(4), P(5), and an arbitrary point on the arc $V(1,2)$ opposite to $V(4,5)$. Let $X_1(5,6)$, $X_1(6,1)$, $X_1(1,2)$, $X_1(2,3)$, $X_1(3,4)$ denote corresponding classes of circles.

Again let $X_2(5)$ denote the class of circles through the fixed point P(5) and any two arbitrary points on $V(1,3)$, composed of $V(1,2)$ and $V(2,3)$. Let $X_2(6)$, $X_2(1)$, $X_2(2)$, $X_2(3)$, $X_2(4)$ denote corresponding classes of circles.

There exists a positive $X_1(4, 5)$-point P(1, 2) on V between P(1) and P(2), as P(1) is positive and therefore P(2) is negative. Similarly there exists a negative $X_1(5, 6)$-point P(2 3) on V between P(2) and P(3). In this way there will exist six points P(1, 2), P(2, 3), P(3, 4), P(4, 5), P(5, 6), P(6, 1) in order on V, alternately positive and negative, which are, respectively, an $X_1(4, 5)$-point, an $X_1(5, 6)$-point, an $X_1(6, 1)$-point, an $X_1(1, 2)$-point, an $X_1(2, 3)$-point and an $X_1(3, 4)$-point. Since P(1, 2) is a positive $X_1(4, 5)$-point and P(2, 3) is a negative $X_1(5, 6)$-point they determine two circles C^2_2 and C^2_3 of the class $X_2(5)$, such that the ranges pertaining to them on $V(12, 23)$ are of contrary signs. Hence there exists a positive $X_2(5)$-point P(123) on V, between P(1, 2) and P(2, 3). Similarly there will exist a negative X_2 (6)-point P(234) on V between P(2, 3) and P(3, 4).

Thus there will exist six points P(123), P(234), P(345), P(456), P(561) and P(612) in order on V, alternately positive and negative, which are, respectively, an $X_2(5)$, an $X_2(6)$, an $X_2(1)$, an $X_2(2)$, an $X_2(3)$ and an $X_2(4)$-point.

Since P(123) is a positive $X_2(5)$-point and P(234) is a negative $X_2(6)$-point they determine two circles C_5^3 and C_6^3 of class X_3, such that the ranges pertaining to them on $V(123, 234)$ are of contrary signs. Hence there exists a positive X_3-point P(1234) on V between P(123) and P(234). Similarly there exists a negative X_3-point P(2345) on V between P(234) and P(345).

Thus there exists six X_3-points P(1234), P(2345), P(3456), P(4561), P(5612) and P(6123) in order on V which are alternately positive and negative. The theorem is therefore proved for $n=3$.

The above proof is quite general in method and can be applied to any value of n.[1]

Cor. The minimum number of cyclic points on an oval is four.

This theorem was first published by me in 1909 in the Bulletin of the Calcutta Mathematical Society for the year.

THEOREM X.

If a conic C intersects an oval V in 2n points ($n>2$) then there exists at least 2n sextactic points, in order on V, which are alternately positive and negative, provided V has continuity of order 5.

To fix ideas suppose $n=4$, and that the intersections taken in order are P(1), P(2),..........,P(8), the point P(1) being a positive intersection.

Let X_5 denote the class of conics through any five arbitrary points of V. Let $X_1(4567)$ denote the class of conics through the four fixed points P(4), P(5), P(6), P(7) and any arbitrary point on V (1, 2). Let $X_2(567)$ denote the class of conics through the three fixed points P(5), P(6), P(7) and any two arbitrary points on $V(1, 3)$.

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[1] *W. Blaschke*, Kreis und Kugel, Leipzig, 1916, p. 161.

Let $X_3(6, 7)$ denote the class of conics through the two fixed points P(6), P(7) and any three arbitrary points on $V(1, 4)$ and finally let $X_4(7)$ denote the class of conics through the fixed point P(7) and any four arbitrary points on $V(1, 5)$.

In general X_{5-r} $(p+1, p+2, \ldots\ldots, p+r)$, where $r=0, 1, 2, 3, 4$ and $p=0, 1, 2, 3, \ldots\ldots, 7$ and $P(q) \equiv P(q-8)$ when q exceeds 8, denotes a class of conics through r fixed points $P(p+1)$, $P(p+2), \ldots\ldots, P(p+r)$ and $5-r$ arbitrary points on $V(p+r+2, p+7)$.

Proceeding as in Theorem IX we have first 8 points P(1, 2), P(2, 3),……,P(8, 1), alternatively positive and negative, which are, respectively, an $X_1(4567)$-point, an $X_1(5678)$-point,……, an X_1 (3456)-point.

Next we have 8 points P(123), P(234),……,P(812), alternately positive and negative, which are, respectively, an $X_2(567)$-point, an $X_2(678)$-point,……an $X_2(456)$-point.

Next we have 8 points P(1234), P(2345),……,P(8123), alternately positive and negative, which, are respectively, an $X_3(67)$-point, an $X_3(78)$-point,……,an $X_3(56)$-point.

Next we have 8 points P(12345), P(23456),……,P(81234), alternately positive and negative, which are respectively, an $X_4(7)$-point, an $X_4(8)$-point,……,an $X_4(6)$-point.

Lastly we have 8 points P(123456), P(234567),……,P(812345), alternately positive and negative, which are all X_5-points.

Thus the theorem is proved when $n=4$. The method is general and applicable to all values of n greater than 2.

Cor. The minimum number of sextactic points on an oval is six.

This theorem was first published by me in the Bulletin of the Calcutta Mathematical Society for the year 1909.

INTRINSIC PARAMETERS IN THE DIFFERENTIAL GEOMETRY OF CURVES IN AN N-SPACE *

BY

S. MUKHOPADHYAYA.

I. GENERAL CONCEPTIONS OF PARAMETRIC CO-EFFICIENTS.

1. Parametric Co-efficients defined in n-dimensional space.

Let a curve in an Euclidean n-dimensional space be defined by

$$x_1=F_1(t),\ x_2=F_2(t),\ x_3=F_3(t), \ldots\ldots\ldots\ldots,\ x_n=F_n(t),$$

where $x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_n$ are the co-ordinates of a point P, on the curve, with reference to n axes mutually orthogonal, through a given origin. Any of the co-ordinates x is the length of perpendicular, with proper sign, on the $(n-1)$ dimensional space passing through the remaining $(n-1)$ axes. The functions $F_1, F_2, F_3, \ldots\ldots$, F_n, as also their derivatives, up to any required order, are supposed to exist and be continuous within the limits of value considered of the parameter t.

If we write

$$\frac{d^r}{dt^r} \equiv D^r, \text{ where } r \text{ is a positive integer,}$$

and

$\Sigma D^{m_1}x_r\, D^{m_2}x_r \equiv (m_1, m_2)$, where $r=1, 2, 3, \ldots\ldots, n$, then (m_1, m_2) will be called a parametric co-efficient of class 1.

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* Abstracted from *Griffith Memorial Prize Essay* for 1910, published by the Calcutta University.

Again, if we write

$$\Sigma \begin{vmatrix} D^{m_1}x_{r_1}, & D^{m_2}x_{r_1} \\ D^{m_1}x_{r_2}, & D^{m_2}x_{r_2} \end{vmatrix}^2 \equiv [m_1, m_2]^2$$

where $r_1 = 1, 2, 3, \ldots, n$

$r_2 = 1, 2, 3, \ldots, n$

then $[m_1, m_2]$ taken positively, will be called a parametric co-efficient of class 2.

Similarly, if we write

$$\Sigma \begin{vmatrix} D^{m_1}x_{r_1}, & D^{m_2}x_{r_1}, & D^{m_3}x_{r_1} \\ D^{m_1}x_{r_2}, & D^{m_2}x_{r_2}, & D^{m_3}x_{r_2} \\ D^{m_1}x_{r_3}, & D^{m_2}x_{r_3}, & D^{m_3}x_{r_3} \end{vmatrix}^2 \equiv [m_1, m_2, m_3]^2$$

where $r_1 = 1, 2, 3 \ldots\ldots\ldots\ldots, n$

$r_2 = 1, 2, 3, \ldots\ldots\ldots\ldots, n$

$r_3 = 1, 2, 3, \ldots\ldots\ldots\ldots, n$

then $[m_1, m_2, m_3]$ taken positively will be called a parametric co-efficient of class 3, and so on.

Finally

$$\begin{vmatrix} D^{m_1}x_1, \ldots, D^{m_n}x_1 \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ D^{m_1}x_n, \ldots, D^{m_n}x_n \end{vmatrix} \equiv [m_1, m_2, \ldots\ldots, m_n]$$

will be called a parametric co-efficient of class n, or of the highest class, for the curve in n-dimensional space.

Further

$$\Sigma \begin{vmatrix} D^{m_1}x_{r_1}, & D^{m_2}x_{r_1} \\ D^{m_1}x_{r_2}, & D^{m_2}x_{r_2} \end{vmatrix} \begin{vmatrix} D^{p_1}x_{r_1}, & D^{p_2}x_{r_1} \\ D^{p_1}x_{r_2}, & D^{p_1}x_{r_2} \end{vmatrix} \equiv [m_1, m_2 \mid p_1, p_2]$$

where

$$r_1 = 1, 2, 3, \ldots\ldots\ldots n$$

$$r_2 = 1, 2, 3, \ldots\ldots\ldots n$$

will be called the moment of the parametric co-efficients

$$[m_1, m_2] \text{ and } [p_1, p_2].$$

Similarly,

$$\Sigma \begin{vmatrix} D^{m_1}x_{r_1}, & D^{m_2}x_{r_1}, & D^{m_3}x_{r_1} \\ D^{m_1}x_{r_2}, & D^{m_2}x_{r_2}, & D^{m_3}x_{r_2} \\ D^{m_1}x_{r_3}, & D^{m_2}x_{r_3}, & D^{m_3}x_{r_3} \end{vmatrix} \begin{vmatrix} D^{p_1}x_{r_1}, & D^{p_2}x_{r_1}, & D^{p_3}x_{r_1} \\ D^{p_1}x_{r_2}, & D^{p_2}x_{r_2}, & D^{p_3}x_{r_2} \\ D^{p_1}x_{r_3}, & D^{p_2}x_{r_3}, & D^{p_3}x_{r_3} \end{vmatrix}$$

$$\equiv [m_1, m_2, m_3 \mid p_1, p_2, p_3]$$

where

$$r_1 = 1, 2, 3, \ldots\ldots, n$$

$$r_2 = 1, 2, 3 \ldots\ldots, n$$

$$r_3 = 1, 2, 3, \ldots\ldots, n$$

will be called the moment of the parametric co-efficients

$$[m_1, m_2, m_3] \text{ and } [p_1, p_2, p_3], \text{ and so on.}$$

We have thus moments of parametric co-efficients, taken in pairs, of any class, from the second upwards. If the co-efficients be identical, then their moment is equal to their product.

Thus,

$$[m_1, m_2 \mid m_1, m_2] = [m_1, m_2]^2$$

$$[m_1, m_2, m_3 \mid m_1, m_2, m_3] = [m_1, m_2, m_3]^2,$$

and so on.

Also the moment of two parametric co-efficients of the highest class n, is the product of the co-efficients.

Thus, $[m_1, m_2, \ldots m_n \mid p_1, p_2, \ldots p_n]$

$$=[m_1, m_2, \ldots, m_n][p_1, p_2, \ldots, p_n]$$

These moments are expressible as simple determinant functions of parametric co-efficients of class 1.

Thus,

$$[m_1, m_2 \mid p_1, p_2]=\begin{vmatrix} (m_1, p_1), (m_2, p_1) \\ (m_1, p_2), (m_2, p_2) \end{vmatrix}$$

$$[m_1, m_2, m_3 \mid p_1, p_2, p_3]=\begin{vmatrix} (m_1, p_1), & m_2, p_1), & (m_3, p_1) \\ (m_1, p_2), & (m_2, p_2), & (m_3, p_2) \\ (m_1, p_3), & (m_2, p_3), & (m_3, p_3) \end{vmatrix}.$$

and so on.

Again, since for space of n dimensions, parametric co-efficients of class higher than n must vanish, we have

$$[m_1, m_2, \ldots\ldots m_n, m_{n+1}]=0, \ [p_1, p_2, \ldots\ldots p_n, p_{n+1}]=0$$

and therefore

$$\begin{vmatrix} (m_1, p_1), \ldots\ldots\ldots\ldots (m_{n+1}, p_1) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (m_1, p_{n+1}), \ldots\ldots\ldots, (m_{n+1}, p_{n+1}) \end{vmatrix}=0$$

2. The n Intrinsic Parameters of a curve in n-dimensional space and their geometrical interpretations.

The first Intrinsic Parameter of a curve in n-dimensional space may be defined as

$$s_1 \equiv \int_{t_0}^{t} (1,1)^{\frac{1}{2}} dt.$$

The second Intrinsic Parameter may be defined as

$$s_2 \equiv \int_{t_0}^{t} [1, 2]^{\frac{1}{2}},$$

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The third Intrinsic Parameter may be defined as

$$s_3 \equiv \int_{t_0}^{t} [1,\ 2,\ 3]^{\frac{1}{6}} dt.$$

The n^{th} Intrinsic Parameter may be defined

$$s_n \equiv \int_{t_0}^{t} [1,\ 2, \ldots n]^{\frac{2}{n(n+1)}} dt.$$

A plane curve has evidently only two Intrinsic Parmeters s_1 and s_2, and a curve in space only three, s_1, s_2 and s_3.

Let P_0 and P be any two points on the curve, corresponding to given values t_0 and t of the parameter t. Take a large number N, of consecutive points on the curve, from P_0 to P, corresponding to N equal small increments δt of t, so that $N\delta t = t - t_0$. Then if $(x_1, x_2 \ldots\ldots, x_n)$ and $(x_1 + \delta x_1,\ x_2 + \delta x_2, \ldots\ldots, x_n + \delta x_n)$ be the co-ordinates of any two consecutive points P_r, P_{r+1}, the length of the chord $P_r P_{r+1}$ is

$$\{(\delta x_1)^2 + (\delta x_2)^2 \ldots\ldots + (\delta x_n)^2\}^{\frac{1}{2}}$$

and the sum of the lengths of N such chords is

$$\Sigma\{(\delta x_1)^2 + (\delta x_2)^2 + \ldots + (\delta x_n)^2\}^{\frac{1}{2}}$$

$$= \Sigma \left\{ \left(\frac{\delta x_1}{\delta t} \right)^2 + \left(\frac{\delta x_2}{\delta t} \right)^2 + \ldots + \left(\frac{\delta x_n}{\delta t} \right)^2 \right\}^{\frac{1}{2}} \delta t .$$

This sum has a limiting value, when N is infinitely large, which may be written as

$$\int_{t_0}^{t} (1,\ 1)^{\frac{1}{2}} dt$$

and which is therefore the first Intrinsic Parameter (s_1). The first Intrinsic Parameter is evidently the same as arc-length (s).

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Again, if P_r, P_{r+1}, P_{r+2} be any three consecutive points on the curve, corresponding to equal small increments δt of t, then if x_1, $x_2, \ldots, x_n$ be the co-ordinates of P_r, those of P_{r+1} and P_{r+2} will be

$$x_1+\delta x_1,\ x_2+\delta_2 x_2 \ldots\ldots\ldots\ldots\ldots\ldots, x_n+\delta x_n,$$

and $$x_1+2\delta_1 x_1+\delta^2 x_1,\ x_2+2\delta x_2+\delta^2 x_2, \ldots\ldots, x_n+2\delta x_n+\delta^2 x_n,$$

respectively.

The projection δS_{12} of the area δS of the triangle $P_r\,P_{r+1}\,P_{r+2}$ on the (x_1, x_2) plane is

$$\frac{1}{2!}\begin{vmatrix} 1, & x_1, & x_2, \\ 1, & x_1+\delta x_1, & x_2+\delta x_2, \\ 1, & x_1+2\delta x_1+\delta^2 x_1, & x_2+2\delta x_2+\delta^2 x_2 \end{vmatrix}$$

$$=\frac{1}{2!}\begin{vmatrix} 1, & 0\ , & 0 \\ 1, & \delta x_1\ , & \delta x_2 \\ -1, & \delta^2 x_1, & \delta^2 x_2 \end{vmatrix}$$

$$=\frac{1}{2!}\left(\delta x_1 \delta^2 x_2-\delta x_2 \delta^2 x_1\right)$$

or

$$\delta S_{12}=\frac{1}{2!}\left(\frac{\delta x_1}{\delta t}\ \frac{\delta^2 x_2}{\delta t^2}-\frac{\delta x_2}{\delta t}\ \frac{\delta^2 x_1}{\delta t^2}\right)\delta t^3.$$

But $$(\delta S)^2=\Sigma(\delta S_{12})^2=\left(\frac{1}{2!}\right)^2\Sigma\left(\frac{\delta x_1}{\delta t}\ \frac{\delta^2 x_2}{\delta t^2}-\frac{\delta^2 x_1}{\delta t^2}\ \frac{\delta^2 x_2}{\delta t^2}\right)^2\delta t^6$$

Therefore

$$\int_{P_0}^{P}(\delta S)^{\frac{1}{3}}=\frac{1}{2!^{\frac{1}{3}}}\int_{t_0}^{t}\left\{\Sigma\left(\frac{dx_1}{dt}\ \frac{d^2 x_2}{dt^2}-\frac{d^2 x_1}{dt^2}\ \frac{dx_2}{dt}\right)^2\right\}^{\frac{1}{6}}dt$$

$$=\frac{1}{(2!)^{\frac{1}{3}}}\int_{t_0}^{t}[1,\ 2]^{\frac{1}{3}}dt.$$

So that the second Intrinsic Parameter

$$s_2=\int_{t_0}^{t}[1,\ 2]^{\frac{1}{3}}dt=(2!)^{\frac{1}{3}}\int_{P_0}^{P}(\delta S)^{\frac{1}{3}}.$$

Similarly, if $P_r,\ P_{r+1},\ P_{r+2},\ P_{r+3}$ be any four consecutive points on the curve, corresponding to equal small increments δt of t, then if $x_1,\ x_2,\dots\dots\dots x_n$ be the co-ordinates of P_r those of $P_{r+1},\ P_{r+2},\ P_{r+3}$ will be respectively

$$x_1+\delta x_1,\qquad x_2+\delta x_2,\dots\dots\dots,x_n+\delta x_n,$$

$$x_1+2\delta x_1+\delta^2 x_1,\quad x_2+2\delta x_2+\delta^2 x_2,\dots x_n+2\delta x_n+\delta^2 x_n,$$

$$x_1+3\delta x_1+3\delta^2 x_1+\delta^3 x_1,\ x_2+3\delta x_2+3\delta^2 x_2+\delta^3 x_2,\dots x_n+3\delta x_n+3\delta^2 x_n$$
$$+\delta^3 x_n.$$

The projection δV_{123} of the volume δV of the tetrahedron $P_rP_{r+1}P_{r+2}P_{r+3}$ on the space $(x_1,\ x_2,\ x_3)$ is

$$\frac{1}{3!}\begin{vmatrix} 1, & x_1, & x_2, & x_3 \\ 1, & x_1+\delta x_1, & x_2+\delta x_2, & x_3+\delta x_3 \\ 1, & x_1+2\delta x_1+\delta^2 x_1, & x_2+2\delta x_2+\delta^2 x_2, & x_3+2\delta x_3+\delta^2 x_3 \\ 1, & x_1+3\delta x_1+3\delta^2 x_1+\delta^3 x_1, & x_2+3\delta x_2+3\delta^2 x_2+\delta^3 x_2, & x_3+3\delta x_3+3\delta^2 x_3+\delta^3 x_3 \end{vmatrix}=\frac{1}{3!}\begin{vmatrix} \delta x_1, & \delta x_2, & \delta x_3 \\ \delta^2 x_1, & \delta^2 x_2, & \delta^2 x_3 \\ \delta^3 x_1, & \delta^3 x_2, & \delta^3 x_3 \end{vmatrix}$$

$$\text{But } (\delta V)^2=\Sigma(\delta V_{123})^2=\frac{1}{(3!)^2}\,\Sigma\begin{vmatrix} \dfrac{\delta x_1}{\delta t}, & \dfrac{\delta x_2}{\delta t}, & \dfrac{\delta x_3}{\delta t} \\ \dfrac{\delta^2 x_1}{\delta t}, & \dfrac{\delta^2 x_2}{\delta t^2}, & \dfrac{\delta^2 x_3}{\delta t^2} \\ \dfrac{\delta^3 x_1}{\delta t^3}, & \dfrac{\delta^3 x_2}{\delta t^3}, & \dfrac{\delta^3 x_3}{\delta t^3} \end{vmatrix}^2(\delta t)^{12}$$

Therefore

$$\int_{P_0}^{P} (\delta V)^{\frac{1}{6}} = \frac{1}{(3!)^{\frac{1}{6}}} \int_{t_0}^{t} \left\{ \Sigma[1, 2, 3]^2 \right\}^{\frac{1}{12}} dt$$

$$= \frac{1}{(3!)^{\frac{1}{6}}} \int_{t_0}^{t} [1, 2, 3]^{\frac{1}{6}}\, dt$$

So that the third Intrinsic Parameter

$$s_3 = (3!)^{\frac{1}{6}} \int_{P_0}^{P} (\delta V)^{\frac{1}{6}}$$

In the same way if we take $p+1$ consecutive points P_r, P_{r+1}, P_{r+2}, ..., P_{r+p} on the curve, the p-dimensional content U_p of the rectilineal figure formed by them is determined by

$$(\delta U_p)^2 = \frac{1}{(p!)^2} \Sigma \begin{vmatrix} \delta x_r \ldots\ldots\ldots\ldots\ldots \delta x_{r+p} \\ \delta^2 x_r \ldots\ldots\ldots\ldots, \delta^2 x_{r+p} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta^p x_r, \ldots\ldots\ldots, \delta^p x_{r+p} \end{vmatrix}^2$$

Therefore

$$\int_{P_0}^{P} (\delta U_p)^{\frac{2}{p(p+1)}} = (p!)^{-\frac{2}{p(p+1)}} \int_{t_0}^{t} \{\Sigma[1, 2, \ldots\ldots p]^2\}^{\frac{1}{p(p+1)}} dt$$

$$= (p!)^{-\frac{2}{p(p+1)}} \int_{t_0}^{t} [1, 2, \ldots\ldots p]^{\frac{2}{p(p+1)}} dt$$

So that if s_p be the p^{th} Intrinsic Parameter

$$s_p = (p!)^{\frac{2}{p(p+1)}} \int_{P_0}^{P} (\delta U_p)^{\frac{2}{p(p+1)}}$$

It may not be superfluous here, to point out that the content of a p-dimensional pyramid, whose vertex is P and base the $(p-1)$-dimensional rectilinear figure determined by $P', P''...P^{(p)}$ is $\frac{1}{p} U_{p-1} H_{p-1}$ where H_{p-1} is height of vertex over base and U_{p-1} the content of the base, for if we take δU_p to be the element of U_p between two parallel $(p-1)$-dimensional bases, separated by distance δH_{p-1} then because U_{p-1} varies as $(H_{p-1})^{p-1}$ we must have

$$U_p = \int_0^{Hp-1} U_{p-1} \, dH_{p-1} = \int_0^{Hp-1} k(\mathrm{H}_{p-1})^{p-1} \, dH_{p-1}$$

$$= \frac{k}{p} H^p{}_{p-1} = \frac{1}{p} H_{p-1} U_{p-1}.$$

Similarly $U_{p-1} = \frac{1}{p-1} H_{p-2} \, U_{p-2}$, and so on.

Therefore $U_p = \frac{1}{p!} H_{p-1} \, H_{p-2} ...... H_1$

whence we deduce in the usual way the formula

$$U_p = \frac{1}{p!} \begin{vmatrix} 1, & x_1, & x_2 & ...... & x_p \\ 1, & x_1', & x'_2 & ...... & x'_p, \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1, & x_1^{(p)}, & x_2^{(p)} & .... & x_p^{(p)} \end{vmatrix}$$

where $x_1, x_2......x_p; x_1', x_2'......x'_p;......; x_1(p), x_2(p)......., x_n(p)$; are the co-ordinates of $p+1$ points $P, P'...P(p)$ in p-dimensional space.

It will appear from the above geometrical interpretations that the n parameters $s_1, s_2, ... s_n$ are intrinsically connected with the curve and give, as it were, its measure in respectively one, two,....n dimensions. The values of s_1, s_2, etc., are independent of the system of axes chosen and of the parameter t and only vary with the positions of P_0 and P on the curve. The n co-ordinates of a point

P on the curve may be expressed as functions of any one of these parameters. Any $n-1$ independent equations between these intrinsic parameters will determine a curve in n-dimensional space, intrinsically.

3. The $n-1$ radii of intrinsic curvature of a curve in n-dimensional space.

If $s_1, s_2, \ldots\ldots, s_n$ be the n intrinsic parameters, then

$$\frac{ds_1}{dt}=(1,\ 1)^{\frac{1}{2}},\ \frac{ds_2}{dt}=[1,\ 2]^{\frac{1}{3}},\ \frac{ds_3}{dt}=[1,\ 2,\ 3]^{\frac{1}{6}},\text{etc.}$$

$$\frac{ds_n}{dt}=[1,\ 2,\ 3\ldots\ldots n]^{\frac{2}{n(n+1)}}$$

If we define the radius of first intrinsic curvature as

$$\rho_1=\left(\frac{ds_1}{ds_2}\right)^3=\frac{(1,\ 1)^{\frac{3}{2}}}{[1,\ 2]},$$

that of second intrinsic curvature as

$$\rho_2=\left(\frac{ds_2}{ds_3}\right)^6=\frac{[1,\ 2]^2}{[1,\ 2,\ 3]}$$

and that of third intrinsic curvature as

$$\rho_3=\left(\frac{ds_3}{ds_4}\right)^{10}=\frac{[1,\ 2,\ 3]^{\frac{5}{3}}}{[1,\ 2,\ 3,\ 4]},$$

and so on.

The radius of $n-1^{th}$ intrinsic curvature is

$$\rho_{n-1}=\left(\frac{ds_{n-1}}{ds_n}\right)^{\frac{n(n+1)}{2}}=\frac{[1,\ 2,\ 3,\ldots\ldots n-1]^{\frac{n+1}{n-1}}}{[1,\ 2,\ 3\ldots\ldots n]}.$$

The dimensions of $\rho_1, \rho_2\ldots\ldots\ldots\rho_{n-1}$ are obviously of the first degree in length. It is easily shewn that the first radius of intrinsic curvature represents the radius of a circle through three consecutive points.

The higher intrinsic parameters are expressible in terms of the first intrinsic parameter and the radii of intrinsic curvature.

For, we have from definitions

$$[1,\,2]^{\frac{1}{3}}=\rho_1^{-\frac{1}{3}}\,(1,\,1)^{\frac{1}{2}},\quad [1,\,2,\,3]^{\frac{1}{6}}=\rho_2^{-\frac{1}{6}}\,[1,\,2]^{\frac{1}{3}},$$

$$[1,\,2,\,3,\,4]^{\frac{1}{10}}=\rho_3^{-\frac{1}{10}}\,[1,\,2,\,3]^{\frac{1}{6}}\ \text{. etc.}$$

$$[1,\,2,\,3,\ldots\ldots n]^{\frac{2}{n(n+1)}}=\rho_{n-1}^{-\frac{2}{n(n+1)}}\,[1,\,2,\,3\ldots\ldots n-1]^{\frac{2}{(n-1)n}}.$$

Therefore

$$s_2=\int_{t_0}^{t}[1,\,2]^{\frac{1}{3}}dt=\int_{t_0}^{t}\rho_1^{-\frac{1}{3}}\frac{ds}{dt}\,dt=\int_{s_0}^{s}\rho_1^{-\frac{1}{3}}ds$$

$$s_3=\int_{t_0}^{t}[1,2,\,3]^{\frac{1}{6}}dt=\int_{s_0}^{s}\rho_1^{-\frac{1}{3}}\,\rho_2^{-\frac{1}{6}}ds$$

... ...

$$s_n=\int_{t_0}^{t}[1,\,2.\,3\ldots\ldots n]^{\frac{2}{n(n+1)}}dt$$

$$=\int_{s_0}^{s}\rho_1^{-\frac{1}{3}}\,\rho_2^{-\frac{1}{6}}\,\rho_3^{-\frac{1}{10}}\ldots\ldots\rho_{n-1}^{-\frac{2}{n(n+1)}}ds$$

The invariant nature of the intrinsic parameters is, therefore, connected with the invariance of $\rho_1,\ \rho_2,\ \rho_3,\ldots\ldots\ldots\rho_{n-1}$.

4. The Spheric of Osculation.

The spheric of $(n-1)$ dimensional boundary which has closest contact with a curve in n-dimensional space has for its equation

$$\begin{vmatrix}
\Sigma(X_r-x_r)^2, & X_1-x_1,\ldots\ldots\ldots\ldots\ldots\ldots X_n-x_n \\
0 \quad , & Dx_1,\ldots\ldots\ldots\ldots\ldots\ldots,\ Dx_n \\
2(1,\,1) \quad , & D^2x_1,\ldots\ldots\ldots\ldots\ldots\ldots,\ D^2x_n \\
6(1,\,2) \quad , & D^3x_1,\ldots\ldots\ldots\ldots\ldots\ldots,\ D^3x_n \\
\ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \\
n(1,\,n-1)+\dfrac{n(n-1)}{2!}(2,\,n-1)+\text{etc.}, & D^nx_1,\ldots..D^nx_n
\end{vmatrix}=0$$

This is deduced from the spheric through $n+1$ points, which has for equation

$$\begin{vmatrix} \Sigma(X_r-x_r)^2, & X_1-x_1, & \ldots\ldots\ldots\ldots & X_n-x_n \\ \Sigma(x_r'-x_r)^2, & x_1'-x_1, & \ldots\ldots\ldots\ldots & x'_n-x_n \\ \ldots & \ldots & \ldots & \ldots \\ \Sigma(x_r^{(n)}-x_r)^2, & x_1^{(n)}-x_1, & \ldots\ldots\ldots\ldots, & x_n^{(n)}-x^n \end{vmatrix}=0$$

or

$$\begin{vmatrix} \Sigma(X_r-x_r)^2, & X_1-x_1, & \ldots\ldots\ldots\ldots \\ \Sigma(x_r'-x_r)^2, & x_1'-x_1, & \ldots\ldots\ldots\ldots \\ \Sigma\{(x_r''-x_r)^2-2(x_r'-x_r)^2\}, & x_1''-2x_1'+x_1, & \ldots\ldots\ldots\ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \Sigma\{(x_r^{(n)}-x_r)^2-n(x_r^{(n-1)}-x_r)^2+\text{etc.}\}, & x_1^{(n)}-nx_1^{(n-1)}+\text{etc.}, & \ldots \end{vmatrix}=0$$

or using the notation of the calculus of Finite Differences

$$\begin{vmatrix} \Sigma(X_r-x_r)^2, & X-x_1, & \ldots\ldots\ldots\ldots & X_n-x_n \\ \Sigma\delta(X_r-x_r)^2, & \delta x_1, & \ldots\ldots\ldots\ldots, & \delta x_n \\ X_r=x_r & & & \\ \Sigma\delta^2(X_r-x_r)^2, & \delta^2 x_1, & \ldots\ldots\ldots\ldots, & \delta^2 x_n \\ X_r=x_r & & & \\ \ldots & \ldots & \ldots & \ldots \\ \Sigma\delta^n(X_r-x^{n}_r)^2, & \delta^n x_1, & \ldots\ldots\ldots\ldots, & \delta^n x_n \\ X_r=x_r & & & \end{vmatrix}=0$$

Now if we divide the second, third, etc., rows by δt, $(\delta t)^2$, $(\delta t)^3$, etc., and go to limits, we have

$$Lt\frac{\delta^n x_r}{(\delta t)^n}=D^n x_r$$

$$\text{and } Lt\frac{\delta^n(X_r-x_r)^2}{(\delta t)^n}=D^n(X_r-x_r)^2_{X_r=x_r}$$

$$X_r=x_r$$

$$=n\,Dx_r\,D^{n-1}x_r+\frac{n(n-1)}{2!}D^2x_rD^{n-2}x_r+\ldots.$$

The equation of the osculating spheric in n-dimensional space can be transformed into

$$\begin{vmatrix} \Sigma(X_r-x_r)^2, & L_1, & L_2, & \ldots\ldots\ldots\ldots & L_n \\ 0 & [1,2,\ldots n], & 0, & \ldots\ldots\ldots\ldots, & 0 \\ 2(1,1), & 0, & [1,2,\ldots n], & \ldots\ldots\ldots, & 0 \\ 6(1,2), & 0, & 0, & \ldots\ldots\ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ n(1,n-1)+\dfrac{n(n-1)}{2!}(2,n-2)+\text{etc.}, & 0, & 0, & \ldots, & [1,2,\ldots n] \end{vmatrix}=0$$

or,

$$[1,2,\ldots, n]\Sigma(X_r-x_r)^2-2(1,1)L_2-6(1,2)L_3-\ldots$$
$$-\left\{n(1,n-1)+\frac{n(n-1)}{2!}(2,n-1)+\text{etc.}\right\}L_n=0.$$

The transforming factor is

$$\begin{vmatrix} 1, & 0, & 0, & \ldots, & 0 \\ 0, & A_{11}, & A_{12}, & \ldots, & A_{1n} \\ 0, & A_{21}, & A_{22}, & \ldots, & A_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & A_{n1}, & A_{n2}, & \ldots, & A_{nn} \end{vmatrix}$$

where $A_{11}, A_{12}, \ldots, A_{nn}$ are the first minors of the determinant

$$[1,2,3,\ldots n]\equiv\begin{vmatrix} Dx_1, & Dx_2, & \ldots, & Dx_n \\ D^2x_1, & D^2x_2, & \ldots, & D^2x_n \\ D^3x_1, & D^3x_2, & \ldots, & D^3x_n \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ D^nx_1, & D^nx_2, & \ldots, & D^nx_n \end{vmatrix}$$

and

$$L_1\equiv(X_1-x_1)A_{11}+(X_2-x_2)A_{12}+\ldots+(X_n-x_n)A_{1n}$$
$$L_2\equiv(X_1-x_1)A_{21}+(X_2-x_2)A_{22}+\ldots+(X_n-x_n)A_{2n}$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$L_n\equiv(X_1-x_1)A_{n1}+(X_2-x_2)A_{n2}+\ldots+(X_n-x_n)A_{nn}$$

In particular the equation of the sphere of closest contact in three-dimensional space has for equation

$$[1,2,3]\Sigma(X-x)^2-2(1,1)\,L_2-6(1,2)\,L_3=0$$

where

$$L_2\equiv-(X-x)[1,3]_{yz}-(Y-y)[1,3]_{zx}-(Z-z)[1,3]_{xy}$$

$$L_3\equiv(X-x)[1,2]_{yz}+(Y-y)[1,2]_{zx}+(Z-z)[1,2]_{xy}.$$

5. The Conicoid of Osculation.

In the same manner the equation of the osculating conicoid to a curve in three-dimensional space can be written as

$$\left|\begin{array}{lllll}
(X-x)^2, & \ldots\ldots\ldots, & (Y-y)(Z-z), & \ldots\ldots\ldots & .(X-x),\ldots \\
0, & \ldots\ldots\ldots, & 0, & \ldots\ldots\ldots, & Dx, \\
2\,DxDx, & \ldots\ldots\ldots, & 2DyDz, & \ldots\ldots\ldots. & D^2x,\ldots \\
6\,DxD^2x, & \ldots\ldots\ldots, & 3DyD^2z+3D^2yDz, & \ldots\ldots, & D^3x,\ldots \\
8\,DxD^3x+6D^2xD^2x, & \ldots & 4DyD^3y+6D^2yD^2y+4D^3yDz, & \ldots & D^4x,\ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
16\,DxD^7x+56D^2xD^6x+\text{etc.}, & & 8DyD^7x+28D^2yD^6z+56D^3yD^5z+\text{etc.}, & \ldots & D^8x,\ldots
\end{array}\right|=0$$

If we multiply this by the transforming factor

$$\left|\begin{array}{llllll}
[2,3]^2_{yz}, & [2,3]^2_{zx}, & [2,3]^2_{xy}, & 2[2,3]_{zx}[2,3]_{xy}, & 2[2,3]_{xy}[2,3]_{yz}, & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
[3,1]_{yz}[1,2]_{yz}, & & \ldots[3,1]_{zx}[1,2]_{xy}+[3,1]_{xy}[1,2]_{zx},\ldots & & & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0, & 0, & 0, & 0, & 0, & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}\right.$$

$$\left.\begin{array}{lllll}
2[2,3]_{yz}[2,3]_{zx}, & 0, & 0, & 0 & \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0, & & [2,3]_{yz}, & [2,3]_{zx}, & [2,3]_{xy} \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array}\right|$$

and write

$$L_{23} \equiv (X-x)[2,3]_{yz} + (Y-y)[2,3]_{zx} + (Z-z)[2,3]_{xy}$$

$$L_{31} \equiv (X-x)[3,1]_{yz} + (Y-y)[3,1]_{zx} + (Z-z)[3,1]_{xy}$$

$$L_{12} \equiv (X-x)[1,2]_{yz} + (X-y)[1,2]_{zx} + (Z-z)[1,2]_{xy}$$

we get, by using the reducing formulae

$$[m,n,p] \equiv [m,n]_{yz}D^px + [m,n]_{zx}D^py + [m,n]_{xy}D^pz$$

$$[m,n,p][r,s,q] + [m,n,q][r,s,p]$$

$$= 2[m,n]_{yz}[r,s]_{yz}D^pxD^qx + \dots$$

$$+ \{[m,n]_{zx}[r,s]_{xy} + [m,n]_{xy}[r,s]_{zx}\}\{D^pyD^qz + D^pzD^qy\} + \dots$$

the following determinant equation

$$\left|\begin{array}{ll}
L_{23}^2 & L_{31}^2 \\
0 & 0 \\
2[1,2,3]^2 & 0 \\
0 & 0 \\
0 & 6[1,2.3]^2 \\
10[1,2,3][2,3,4] & 0 \\
12[1,2,4][2,3,5] & -30[1,2,3][1,3,4] \\
14[1,2,3][2,3.6] & -42[1,2,3][1,3,5] \\
16[1,2,3][2,3,7]+70[2,3,4]^2, & -56[1,2,3][1,3,6]+70[1,3,4]^2,
\end{array}\right.$$

$$\begin{array}{l}
L_{12}^2 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}$$

$20[1, 2, 3]^2$

$70[1, 2, 3][1, 2, 4]$

$112[1, 2, 3][1, 2, 5]+70[1, 2, 4]^2$

$L_{23} L_{31}$

0

0

$3[1, 2, 3]^2$

0

$-5[1, 2, 3][1, 3, 4]$

$-6[1, 2, 3][1, 3, 5]+15[1, 2, 3][2, 3, 4]$

$-7[1, 2, 3][1, 3, 6]+21[1, 2, 3][2, 3, 5]$

$-8[1, 2, 3][1, 3, 7]+28[1, 2, 3][2, 3, 6]-70[1, 3, 4][2, 3, 4],$

$L_{31} L_{12}$

0

0

0

0

$10[1, 2, 3]^2$

$15[1, 2, 3][1, 2, 4]$

$21[1, 2, 3][1, 2, 5]-35[1, 2, 3][1, 3, 4]$

$28[1, 2, 3][1, 2, 6]-56[1, 2, 3][1, 3, 5]-70[1, 2, 4][1, 3, 4],$

$L_{12} L_{23}$

0

0

0

$4[1, 2, 3]^2$

$5[1, 2, 3][1, 2, 4]$

$6[1, 2, 3][1, 2, 5]$

$7[1, 2, 3][1, 2, 6]+35[1, 2, 3][2, 3, 4]$

$8[1, 2, 3][1, 2, 7]+56[1, 2, 3][2, 3, 5]+70[1, 2, 4][2, 3, 4].$

$$\left.\begin{array}{ccc}
L_{23} & L_{31} & L_{12} \\
[1,2,3] & 0 & 0 \\
0 & [1,2,3] & 0 \\
0 & 0 & [1,2,3] \\
[2,3,4], & -[1,3,4], & [1,2,4] \\
[2,3,5], & -[1,3,5], & [1,2,5] \\
[2,3,6], & -[1,3,6], & [1,2,6] \\
[2,3,7], & -[1,3,7], & [1,2,7] \\
[2,3,8], & -[1,3,8], & [1,2,8]
\end{array}\right| = 0$$

To simplify this equation we may take the third intrinsic parameter s_3 to be the independent variable t. Then because

$$\frac{ds_3}{dt} = [1, 2, 3]^{\frac{1}{6}}$$

we have

$$[1, 2, 3] = 1 \quad D[1, 2, 3] = [1, 2, 4] = 0$$

If we call $[1, 3, 4] = -I$ and $[2, 3, 4] = -J$ all the other parametric co-efficients of class 3 can be calculated in terms of I and J, and their derivatives with respect to s_3.

The formulae for calculation of the parametric co-efficients of class three are

$$[1, 2, n+1] = D[1, 2, n] - [1, 3, n]$$

$$[2, 3, n+1] = D[2, 3, n] - [2, 4, n]$$

$$[1, 3, n+1] = D[1, 3, n] - [2, 3, n] - [1, 4, n]$$

and

$$[1, 4, n] = \frac{[1, 2, 4]\,[1, 3, n] - [1, 3, 4]\,[1, 2, n]}{[1, 2, 3]}$$

$$[2, 4, n] = \frac{[1, 2, 4]\,[2, 3, n] - [2, 3, 4]\,[1, 2, n]}{[1, 2, 3]}$$

which are particular cases of the general formula

$$-[l, m, n]\,[1, 2, 3]^2 = \begin{vmatrix} [2, 3, l\,], & [1, 3, l\,], & [1, 2, l\,] \\ [2, 3, m], & [1, 3, m], & [1, 2, m] \\ [2, 3, n\,], & [1, 3, n\,], & [1, 2, n\,] \end{vmatrix}$$

If the independent variable be the third intrinsic parameter s_3, then

$$[1, 2, 3]=1, \quad [1, 2, 4]=0, \quad [1, 3, 4]=-I$$

$$[2, 3, 4]=-J, \text{ therefore } [1, 4, n]=I\,[1, 2, n]$$

$$\text{and } [2, 4, n]=J\,[1, 2, n].$$

By the above formulae we obtain the following table of values of the parametric co-efficients of class 3 :—

$[1, 2, 3]=1 \qquad [1, 2, 5]=I \qquad [1, 2, 6]=2I'-J$

$[1, 2, 4]=0 \qquad [2, 3, 5]=J' \qquad [2, 3, 6]=-J''-IJ$

$[1, 3, 4]=-I \qquad [1, 3, 5]=-I'+J \qquad [1, 3, 6]=-I''-I^2+2J'$

$[2, 3, 4]=-J \qquad [1, 4, 5]=I^2 \qquad [1, 4, 6]=2\,II'-IJ$

$[2, 4, 5]=IJ \qquad [2, 4, 6]=2\,I'J-J^2$

$[1, 2, 7]=3\,I''-3\,J'+I^2$

$[2, 3, 7]=-J'''-3\,I'J-IJ'+J^2$

$[1, 3, 7]=-I'''-4\,II'+3J''+2IJ$

$[1, 4, 7]=3II''-3\,IJ'+I^3$

$[2, 4, 7]=3\,I''J-3\,JJ'+I^2J$

$[1, 2, 8]=4\,I'''-6\,J''+6\,II'-2\,IJ$

$[2, 3, 8]=-J^{iv}-6\,I''J-4\,I'J'-IJ''+5JJ'-I^2J$

$[1, 3, 8]=-I^{iv}+4\,J'''-4I'^2-7\,II''+5\,I'J+6\,IJ'-J^2-I^3$

$[1, 4, 8]=4\,I'''I-6\,J''I+6\,I^2I'-2\,I^2J$

$[2, 4, 8]=4\,I'''J-6\,J''J+6\,II'J-2\,IJ^2$

and so on,

The equation of the osculating conicoid is simplified if we take s_3 as the independent variable and substitute the values of the parametric co-efficients from the above table.

II. Application to Plane Curves.

1. Definitions and General Relations.

Suppose x, y are the co-ordinates of a point P of a plane curve defined by

$$x=F_1(t) \text{ and } y=F_2(t),$$

where F_1 and F_2 are given functions of the parameter t.

Then, if $D^n x$ and $D^n y$ be the n^{th} derivatives of x and y with respect to t,

$$D^m x\, D^n x+D^m y\, D^n y\equiv(m,\ n)$$

and

$$D^m x\, D^n y-D^n x\, D^m y\equiv[m,\ n]$$

where (m, n) and $[m, n]$ are parametric co-efficients of classes 1 and 2, respectively. We have $(m, n)=(n, m)$ and $(m, m)=0$.

$$\text{Also } [m,\ n]\ [p,\ q]=\begin{vmatrix} (m,\ p), & (m,\ q) \\ (n,\ p), & (n,\ q) \end{vmatrix}$$

Whence $[m, n]\ [1, 2]=(1, m)\ (2, n)-(1, n)\ (2, m)$ and $[m, n]^2 =(m, m)\ (n, n)-(m, n)^2$.

If we multiply together the matrices

$$\begin{vmatrix} D^l x, & D^l y \\ D^m x, & D^m y \\ D^n x, & D^n y \end{vmatrix} \text{ and } \begin{vmatrix} D^p x, & D^p y \\ D^q x, & D^q y \\ D^r x, & D^r y \end{vmatrix} \text{ we get}$$

$$\begin{vmatrix} (l,\ p), & (l,\ q), & (l,\ r) \\ (m,\ p), & (m,\ q), & (m,\ r) \\ (n,\ p), & (n,\ q), & (n,\ r) \end{vmatrix}=0.$$

From which, putting $m=1,\ n=2,\ q=1,\ r=2$

we get

$$(l,\ p)\ [1,\ 2]^2=(1,\ l)(1,\ p)\ (2,\ 2)+(2,\ l)\ (2,\ p)\ (1,\ 1)$$

$$-\{(1,\ l)\ (2,\ p)+(2,\ l)\ (1,\ p)\}\ (1,\ 2).$$

Also by expanding the determinant we have,

$$(l,\ p)\ [m,\ n]\ [q,\ r]+(m,\ p)\ [n,\ l]\ [q,\ r]+(n,\ p)\ [l,\ m]\ [q,\ r]=0$$

or, $\quad (l,\ p)\ [m,\ n]+(m,\ p)\ [n,\ l]+(n,\ p)\ [l,\ m]=0$

whence $\quad (l,\ p)\ [1,\ 2]=[1,\ l]\ (2,\ p)-[2,\ l]\ (1,\ p)$

and $\quad (1,\ 1)\ [m,\ n]=[1,\ n]\ (1,\ m)-[1,m]\ (1,\ n).$

If we multiply together the matrices

$$\begin{vmatrix} D^l x, -D^l y \\ D^m x, -D^m y \\ D^n y,\ \ D^n x \end{vmatrix} \text{ and } \begin{vmatrix} D^p y,\ D^p x \\ D^q y,\ D^q x \\ D^r y,\ D^r x \end{vmatrix} \text{ we get}$$

$$\begin{vmatrix} [l,\ p], & [l,\ q], & [l,\ r] \\ [m,\ p], & [m,\ q], & [m,\ n] \\ (n,\ p), & (n,\ q), & (n,\ r) \end{vmatrix}=0$$

and if we multiply together the matrices

$$\begin{vmatrix} -D^l y,\ D^l x \\ D^m x,\ D^m y \\ D^n x,\ D^n y \end{vmatrix} \text{ and } \begin{vmatrix} D^p x,\ D^p y \\ D^q x,\ D^q y \\ D^r x,\ D^r y \end{vmatrix} \text{, we get}$$

$$\begin{vmatrix} [l,\ p\ ], & [l,\ q\ ], & [l,\ r,\ ] \\ (m,\ p\ ), & (m,\ q\ ), & (m,\ n,\ ) \\ (n,\ p\ ), & (n,\ q\ ), & (n,\ r,\ ) \end{vmatrix}=0$$

where, if we put $m=1,\ n=2,\ q=1,\ r=2$ we get

$$[l,\ p]\ [1,\ 2]^2=\{[1,\ l]\ (2,p)+[2,l]\ (1,\ p)\}\ (1,\ 2)$$

$$-[1,\ l]\ (1,\ p)\ (2,\ 2)-[2,\ l]\ (2,\ p)\ (1,\ 1).$$

Also, by expanding the determinant, we have

$$[l, p]\,[m, n]\,[q, r]+[l, q]\,[m, n]\,[r, p]+[l, r]\,[m, n]\,[p, q]=0$$

or $$[l, p]\,[q, r]+[l, q]\,[r, p]+[l, r]\,[p, q]=0$$

whence, $$[l, p]\,[1, 2]=[1, l]\,[2, p]-[1, p]\,[2, l].$$

If we multiply together the matrices

$$\begin{vmatrix} -D^l y, & D^l x \\ D^m x, & D^m y \\ D^n x, & D^n y \end{vmatrix} \text{ and } \begin{vmatrix} -D^p y, & D^p x \\ D^q x, & D^q y \\ D^r x, & D^r y \end{vmatrix} \text{ we get}$$

$$\begin{vmatrix} -(l, & p), & [l, & q], & [l, & r] \\ [m, & p], & (m, & q), & (m, & r) \\ [n, & p], & (n, & q), & (n, & r) \end{vmatrix}=0$$

where if we put $m=1$, $n=2$, $q=1$, $r=2$, we get

$$(l, p)\,[1, 2]^2=[1, l]\,[1, p]\,(2, 2)+[2, l]\,[2, p]\,(1, 1)$$
$$-\{[1, l]\,(2, p)+[2, l]\,[1, p]\}\,(1, 2).$$

Finally if we multiply together the matrices

$$\begin{vmatrix} -D^l y, & D^l x \\ D^m x, & D^m y \\ D^n x, & D_n y \end{vmatrix} \text{ and } \begin{vmatrix} D^p x, & D^p y \\ -D^q y, & D^q x \\ -D^r y, & D^r x \end{vmatrix}\text{, we get}$$

$$\begin{vmatrix} -[l, p], & (l, q), & (l, r) \\ (m, p), & [m, q], & [m, r] \\ (m, p), & [n, q], & [n, r] \end{vmatrix}$$

where if we put $m=1$, $n=2$, $q=1$, $r=2$, we get

$$[l, p]\,[1, 2]=(1, l)\,(2, p)-(1, p)\,(2, l)$$

a relation otherwise evident.

2. Working Formulae.

All the other parametric co-efficients can be easily calculated, if we know (1, 1) and (2, 2), and therefore (1, 2), (1, 3), [1, 2], (2, 3) and [2, 3].

$$\text{For } D(1,1)=2(1,2),\ [1,2]^2=(1,1)(2,2)-(1,2)^2,$$

$$D(2,2)=2(2,3),\ D(1,2)=(1,3)+(2,2)$$

$$\text{and } [1,2]\,[2,3]=(1,2)(2,3)-(1,3)(2,2).$$

The following set of formulae, which we may call the first set, will be found useful for general purposes.

Suppose we have calculated $(1, n)$ and $[1, n]$ for all values of n from 1 to n; then we can calculate (r, s) and $[r, s]$, for values of r and s not exceeding n, from

$$(r,s)(1,1)=(1,r)(1,s)+[1,r]\,[1,s]$$

and

$$[r,s](1,1)=(1,r)\,[1,s]-(1,s)\,[1,r].$$

We can next calculate $(1, n+1)$ and $[1, n+1]$ from

$$(1,n+l)=D(1,n)-(2,n)$$

$$[1,n+l]=D[1,n]-[2,n]$$

where

$$(2,n)=\{(1,2)(1,n)+[1,2]\,[1,n]\}/(1,1)$$

and

$$[2,n]=\{(1,2)\,[1,n]-[1,2]\,(1,n)\}/(1,1)$$

The following set of formulae, which we may call the second set, are also useful, and specially so, in certain cases.

Suppose we have calculated $[1, n]$ and $[2, n]$ for all values of n from 1 to n. Then we can calculate $[r, s]$ for values of r and s not exceeding n, from

$$[r,s]=\{[1,r]\,[2,s]-[2,r]\,[1,s]\}/[1,2].$$

We can next calculate $[1, n+1]$ and $[2, n+1]$

from

$$[1,n+1]=D[1,n]-[2,n]$$

and

$$[2,n+1]=D[2,n]-[3,n]$$

where

$$[3, n]=\{[1, 3]\ [2, n]-[2, 3]\ [1, n]\}/[1, 2].$$

Subsequently we can calculate $(1, n)$ $(2, n)$ and (r, s) from

$$(1, n)\ [1, 2]=(1, 2)\ [1, n]-(1, 1)\ [2, n]$$

$$(2, n)\ [1, 2]=(2, 2)\ [1, n]-(1, 2)\ [2, n]$$

and

$$[1, 2]^2(r, s)=[1, r]\ [1, s]\ (2, 2)+[2, r]\ [2, s]\ (1, 1)$$
$$-\{[1, r]\ [2, s]+[1, s]\ [2, r]\}\ (1, 2).$$

The following set of formulae, which we may call the third set, may also be used.

Suppose we have calculated $(1, n)$ and $(2, n)$ for all values of n from 1 to n. Then we can calculate (r, s) and $[r, s]$ for values of r and s not exceeding n, from

$$[1, 2]^2(r, s)=(1, r)\ (1, s)\ (2, 2)+(2, r)\ (2, s)\ (1, 1)$$
$$-\{(1, r)\ (2, s)+(1, s)\ (2, r)\}\ (1, 2)$$

and $$[r, s]=\{(1, r)\ (2, s)-(1, s)\ (2, r)\}/[1, 2].$$

We can next calculate $(1, n+1)$ and $(2, n+1)$, from

$$(1, n+1)=D(1, n)-(2, n)$$

and $$(2, n+1)=D(2, n)-(3, n)$$

where

$$(3, n)=\frac{[1, 3](2, n)-[2, 3](1, n)}{[1, 2]}$$

3. When the first intrinsic parameter (s_1) is the independent variable *t*.

In this case, since $(1, 1)^{\frac{1}{2}}=\frac{ds_1}{dt}=1$, we have $(1, 1)=1$,

$(1, 2)=0$ for $D(1, 1)=2(1, 2)$.

Also $[1, 2]=r$, where $r=\frac{[1, 2]}{(1, 1)^{\frac{3}{2}}}=\frac{1}{\rho}$

Now, if we use the first set of formulae, we get

$$(2,n)=\{(1,2)(1,\ n)+[1,\ 2][1,\ n]\}/(1,\ 1)=r[1,n]$$

$$[2,\ n]=\{(1,\ 2)\ [1,\ n]-[1,\ 2]1,\ n)\}/(1,\ 1)=-r(1,\ n).$$

Therefore

$$(2,\ 2)=r^2,\ (1,\ 3)=D(1,\ 2)-(2,\ 2)=-r^2$$

$$(2,\ 3)=rr',\ \text{for } D(2,\ 2)=2(2,\ 3)$$

$$[1,\ 3]=D[1,\ 2]=r',\ [2,\ 3]=-r(1,\ 3)=r^3$$

$$(3,\ n)=\{(1,\ 3)(1,\ n)+[1,\ 3][1,\ n]\}/(1,\ 1)=-r^2(1,\ n)+r'[1,\ n]$$

$$[3,\ n]=\{(1,\ 3)\ [1,\ n]-[1,\ 3]\ (1,\ n)\}/(1,\ 1)=-r^2[1,\ n]-r'(1,\ n)$$

so that we can at once write down the series $(2,\ n)$, $[2,\ n]$, $(3,\ n)$, $[3,\ n]$ if we calculate the series $(1,\ n)$ and $[1,\ n]$.

The values of the series $(1,\ n)$, $[1,\ n]$, calculated from the first set of formulae, are

$$(1,\ 1)=1,\ [1,\ 2]=r,\ (1,\ 2)=0$$

$$(1,\ 3)=-r^2,\ [1,\ 3]=r'$$

$$(1,\ 4)=-3rr',\ [1,\ 4]=r''-r^3$$

$$(1,\ 5)=-4rr''-3r'^2+r^4,\ [1,\ 5]=r'''-6r^2r'$$

$$(1,\ 6)=-5rr'''-10r'r''+10r^3r'$$

$$[1,\ 6]=r^{iv}-10r^2r''-15rr'^2+r^5$$

$$(1,\ 7)=-6rr^{iv}-15r'r'''-10r''^2+20r^3r''+45r^2r'^2-r^6$$

$$[1,\ 7]=r^{v}-15r^2r'''-60rr'r''+15r^4r'-15r'^3$$

$$(1,\ 8)=-7rr^{v}-21r'r^{iv}-35r''r'''+35r^3r'''+210r^2r'r''+105rr'^3-21r^5r'$$

$$[1,\ 8]=r^{vi}-21r^2r^{iv}-105rr'r'''-70rr''^2-105r'^2r''+35r^4r''+$$
$$105r^3r'^2-r^7$$

$$(1,\ 9)=-8rr^{vi}-28r'r^{v}-56r''r^{iv}+56r^3r^{iv}-35r'''^2+420r^2r'r'''+$$
$$280r^2r''^2+840rr'^2r''-56r^5r''+105r'^4-210r^4r'^2+r^8$$

$$[1,\ 9]=r^{vii}-28r^2r^{v}-168rr'r^{iv}-280rr''r'''-210r'^2r'''-210r'r''^2$$
$$+70r^4r'''+560r^3r'r''+420r^2r'^3-28r^6r',\ \text{and so on.}$$

4. When the second intrinsic parameter s_2 is taken as the independent variable t.

In this case, since

$$[1,\ 2]^{\frac{1}{3}}=\frac{ds_2}{dt}=1, \text{ we have } [1,\ 2]=1 \text{ and } [1,\ 3]=D[1,\ 2]=0.$$

Also, since $(1,\ 1)^{\frac{3}{2}}/[1,\ 2]=\rho$, where ρ is radius of curvature, we have $(1,\ 1)=\rho^{\frac{2}{3}}$.

Again, since $\{3[1,\ 2]\,(1,\ 2)-[1,\ 3](1,\ 1)\}/3[1,\ 2]^2=\tan\delta$, where δ is the angle of aberrancy (*vide* A General Theory of Osculating Conics, Second Paper), we have $(1,\ 2)=\tan\delta$, and because $(1,\ 1)(2,\ 2)=[1,\ 2]^2+(1,\ 2)^2$, we have $(2,\ 2)=(\tan^2\delta+1)\rho^{-\frac{2}{3}}=\sec^2\delta\rho^{-\frac{2}{3}}$.

Also because $\{3[1,\ 2][1,\ 4]-5[1,\ 3]^2+12[1,\ 2][2,\ 3]\}/9[1,\ 2]^{\frac{8}{3}}=(ab)^{-\frac{2}{3}}$, where a, b are the semi-axes of the osculating conic (*vide* A General Theory of Osculating Conics, Second Paper), we have

$$3[1,\ 4]+12[2,\ 3]=9(ab)^{-\frac{2}{3}}$$

$$\text{But}\ [1,\ 4]+[2,\ 3]=D[1,\ 3]=0$$

$$\text{therefore}\ [2,\ 3]=-[1,\ 4]=(ab)^{-\frac{2}{3}}=I, \text{ suppose.}$$

If we use the second set of formulae we have

$$[3,\ n]=\{[1,\ 3][2,\ n]-[2,\ 3][1,\ n]\}/[1,\ 2]=-I[1,n]$$

Therefore, starting from $[1,\ 2]=1$, $[1,\ 3]=0$, $[2,\ 3]=I$, $[1,\ 4]=-I$, we can calculate all the parametric co-efficients of class 2.

The parametric co-efficients of class 1 are then determined from

$$(1,\ n)=\{(1,\ 2)[1,\ n]-(1,\ 1)\,[2,\ n]\}/[1,\ 2]$$

$$(2,\ n)=\{(2,\ 2)\,[1,\ n]-(1,\ 2)[2,\ n]\}/[1,\ 2]$$

and

$$[1,\ 2]^2(r,\ s)=[1,\ r][1,\ s]\,(2,\ 2)+[2,\ r][2,\ s]\,(1,\ 1)$$
$$-\{[1,\ r][2,\ s]+[1,\ s][2,\ r]\}\,(1,\ 2)$$

which give

$$(1,\ n)=[1,\ n]\tan\delta-[2,\ n]\rho^{\frac{2}{3}}$$

$$(2,\ n)=[1,\ n]\sec^2\delta\,\rho^{-\frac{2}{3}}-[2,\ n]\tan\delta$$

and

$$(r,\ s)=[1,\ r][1,\ s]\sec^2\delta\,\rho^{-\frac{2}{3}}-[2,\ r][2,\ s]\,\rho^{\frac{2}{3}}$$
$$-\{[1,\ r][2,\ s]+[1,\ s][2,\ r]\}\tan\delta.$$

The values of the series $[1, n]$, $[2, n]$, calculated from the second set of formulae and expressed in terms of I and its derivatives with respect to s_2, are given in the following table which may be easily extended as far as one wishes. The series $[3, n]$ is at once obtained from the formula

$$[3, n]=-I\ [1, n]$$

and the series $[r, n]$ can be calculated from

$$[r, n]=\{[1, r]\ [2, n]-[2, r]\ [1, n]\}/[1, 2].$$

$[1, 2]=1$, $[1,3]=0$, $[2, 3]=I$

$[1, 4]=-I$, $[2, 4]=I'$

$[1, 5]=-2I'$, $[2, 5]=I''-I^2$

$[1, 6]=-3I''+I^2$, $[2, 6]=I'''-4\ II'$

$[1, 7]=-4I'''+6II'$, $[2, 7]=I^{iv}-7\ II''-4\ I'^2+I^3$

$[1, 8]=-5\ I^{iv}+13\ II''+10\ I'^2-I^3$

$[2, 8]=I^{v}-11\ II'''-15\ I'\ I''+9\ I^2\ I'$

$[1, 9]=-6\ I^{v}+24\ II'''+48\ I'\ I''-12\ I^2\ I'$

$[2, 9]=I^{vi}-16\ II^{iv}-26\ I'\ I'''-15\ I''^2+22\ I^2\ I''+28\ II'^2-I^4$

and so on.

5. Expressions for the length of a Chord.

Let P_o and P_1, be two points on the curve corresponding to values o and t of the parameter. Let the co-ordinates of P_o and P_1 be (x_o, y_o) and (x_1, y_1).

Then, evidently

$$(x_1-x_o)\ Dx+\ (y_1-y_o)Dy$$

$$=(1, 1)\ t+(1,2)\ t^2/2!+(1, 3)\ t^3/3!+(1, 4)\ t^4/4!+\text{etc.}$$

$$\text{and}\ (y_1-y_o)\ Dx-(x_1-x_o)\ Dy$$

$$=[1, 2]\ t^2/2!+[1, 3]\ t^3/3!+[1, 4]\ t^4/4!+\text{etc.}$$

Similarly, if (x_1', y_1') and (x_o, y_o) be the co-ordinates of two points P'_1 and P_o on the curve corresponding to values $-t$ and o of the parameter, then

$$(x_1'-x_o)Dx+(y_1'-y_o)Dy$$

$$=-(1,1)\,t+(1,2)\,t^2/2!-(1,3)\,t^3/3!+(1,4)\,t^4/4!-\text{etc.}$$

and

$$(y_1'-y_o)Dx-(x_1'-x_o)Dy$$

$$=[1,2]\,t^2/2!-[1,3]\,t^3/3!+[1,4]t^4/4!-\text{etc.}$$

Therefore

$$\tfrac{1}{2}\{(x_1-x_1')\,Dx+(y_1-y_1')\,Dy\}$$

$$=(1,1)\,t+(1,3)\,t^3/3!+(1,5)\,t^5/5!+\text{etc.}$$

and

$$\tfrac{1}{2}\{(y_1-y_1')Dx-(x_1-x_1')Dy\}$$

$$=[1,3]\,t^3/3!+[1,5]t^5/5!+\text{etc.}$$

If L be the length of the chord $P_1\,P_1'$ then

$$(1,1)\,L^2=\{(x_1-x_1')\,Dx+(y_1-y_1')Dy\}^2$$

$$+\{(y_1-y_1')Dx-(x_1-x_1')Dy\}^2$$

whence, after simplifications, we have

$$L^2/4\,t^2=(1,1)+2(1,3)t^2/3!+2(1,5)\,t^4/5!+(3,3)t^4/3!\,3!$$

$$+2(1,7)\,t^6/7!+2(3,5)\,t^6/3!\,5!+2\,(1,9)\,t^8/9!$$

$$+2(3,7)\,t^8/3!\,7!+(5,5)\,t^8/5!\,5!+\text{etc.}$$

Whence again, after extraction of square root and simplifications, we have

$$L/2\,t\,(1,1)^{\frac{1}{2}}=1+\frac{t^2}{(1,1)}\,\frac{(1,3)}{3!}+\frac{t^4}{(1,1)^2}\left\{\frac{(1,5)(1,1)}{5!}+\frac{1}{2}\frac{[1,3]^2}{3!3!}\right\}$$

$$+\frac{t^6}{(1,1)^3}\left\{\frac{(1,7)(1,1)^2}{7!}+\frac{[1,3][1,5](1,1)}{3!5!}-\frac{1}{2}\frac{(1,3)[1,3]^2}{3!3!3!}\right\}$$

$$+\frac{t^8}{(1,1)^4}\left\{\frac{(1,9)(1,1)^3}{9!}+\frac{1}{2}\frac{(1,1)^2[1,5]^2}{5!5!}-\frac{1}{2}\frac{(1,5)[1,3]^2(1,1)}{3!5!3!}\right.$$

$$-\frac{(1,3)[1,3][1,5](1,1)}{5!3!3!}+\frac{[1,3][1,7](1,1)^2}{3!7!}+\frac{1}{2}\frac{(1,3)^2[1,3]^2}{3!3!3!3!}$$

$$\left.-\frac{1}{8}\frac{[1,3]^4}{3!3!3!3!}\right\}+\text{etc.}$$

If the independent variable (t) be the second intrinsic parameter s_2, then $[1, 3]=0$, therefore the expression for L reduces to

$$L/2\,t\,(1,1)^{\frac{1}{2}}=1+\frac{t^2}{(1,1)}\frac{(1,3)}{3!}+\frac{t^4}{(1,1)}\frac{(1,5)}{5!}+\frac{t^6}{(1,1)}\frac{(1,7)}{7!}+\text{etc.}$$

or

$$L=2\,s_2\,\rho^{\frac{1}{3}}\left\{1-\frac{s_2^{\,2}}{3!}I-\frac{s_2^{\,4}}{5!}\frac{2\,I'\tan\delta-\rho^{\frac{1}{3}}(I''-I^2)}{\rho^{\frac{2}{3}}}+\text{etc.}\right\}$$

where $2s_2$ is the length of the second intrinsic parameter from P_1' to P_1, the initial point P_0 from which s_2 is measured being so situated as to bisect $2s_2$.

If the independent variable be the first intrinsic parameter s_1, namely, arc-length, then the expression for L becomes

$$L/2s_1=1-s_1^{\,2}r^2/6+s_1^{\,4}\,(3r^4-4r'^2-12rr'')/360+\text{etc.}$$

If we write s for the entire arc $P_1'P_1$ which is $2s_1$, we have

$$L=s-s^3r^2/24+s^5\,(3r^4-4r'^2-12rr'')/5760+\text{etc.}$$

If we shift the origin of s to any arbitrary point on the arc so that the arc distances to P_1' and P_1 are s_1' and s_1, then

$$s=s_1-s_1'$$

and the arc distance of the old origin from the new is $(s_1+s_1')/2$. Therefore

$$L=(s_1-s_1')-\frac{(s_1-s_1')^3}{24}\left\{r^2+\frac{s_1+s_1'}{2}D(r^2)\right.$$

$$\left.+\left(\frac{s_1+s_1'}{2}\right)^2D^2(r^2)/2+\left(\frac{s_1+s_1'}{2}\right)^3D^3(r^2)/3!+\text{etc.}\right\}$$

$$+\frac{(s_1-s_1')^5}{5760}\left\{(3\,r^4-4\,r'^2-12\,rr'')\right.$$

$$\left.+\frac{s_1+s_1'}{2}D(3r^4-4\,r'^2-12\,rr'')+\text{etc.}\right\}+\text{etc.}$$

Note.—The expression for L in terms of the arc is calculated in some text books (*vide* Calcul Differentiel par J. Bertrand) to a few terms, but the method is less general.

6. The Osculating Cubic.

Let x, y be a given point on a curve and X, Y another point so that the value of the second intrinsic parameter from the first to the second point is s_2. Then, if we write

$$L_1 \equiv (Y-y)\frac{dx}{ds_2}-(X-x)\frac{dy}{ds_2}$$

and

$$L_2 \equiv (Y-y)\frac{d^2x}{ds_2^2}-(X-x)\frac{d^2y}{ds_2^2}$$

we have

$$L_1 \equiv [1,2]\frac{s_2^2}{2!}+[1,3]\frac{s_2^3}{3!}+[1,4]\frac{s_2^4}{4!}+\text{etc.}$$

$$L_2 \equiv -[1,2]s_2+[2,3]\frac{s_2^3}{3!}+[2,4]\frac{s_2^4}{4!}+\text{etc.}$$

or,

$$L_1 = s_2^2/2!-Is_2^4/4!-2I's_2^5/5!-(3I''-I^2)s_2^6/6!$$
$$-(4I'''-6II')s_2^7/7!-(5I^{iv}-13II''-10I'^2+I^3)s_2^3/8!$$
$$-(6I^{v}-24II'''-48I'I''+12I^2I')s_2^9/9!-\text{etc.}$$

and

$$L_2 = -s_2+Is_2^3/3!+I's_2^4/4!+(I''-I^2)s_2^5/5!$$
$$+(I'''-4II')s_2^6/6!+(I^{iv}-7II''-4I'^2+I^3)s_2^7/7!$$
$$+(I^{v}-11II'''-15I'I''+9I^2I')\,s_2^8/8!+(I^{vi}-16II^{iv}$$
$$-26I'I'''-15I''^2+22I^2I''+28II'^2-I^4)s_2^9/9!+\text{etc.}$$

Whence,

$$L_1^2 = 6s_2^4/4!-30Is_2^6/6!-84I's_2^7/7!$$
$$-(168I''-126I^2)s_2^8/8!-(288I'''-936II')s_2^9/9!+\text{etc.}$$

$$L_1^3 = 90s_2^6/6!-1260Is_2^8/8!-4536I's_2^9/9!+\text{etc.}$$

$$L_2^2 = 2s_2^2/2!-8Is_2^4/4!-10I's_2^5/5!-(12I''-32I^2)s_2^6/6!$$
$$-(14I'''-126II')s_2^7/7!-(16I^{iv}-224II''-134I'^2+$$
$$128I^3)s_2^8/8!$$
$$-(18I^{v}-366II'''-522I'I''+1086I^2I')s_2^9/9!-\text{etc.}$$

$$S \equiv L_2^2-2L_1+IL_1^2=-6I's_2^5\,/5!-6I''s_2^6/6!-(6I'''-$$
$$30I^{I'})s_2^7/7!$$
$$-(6I^{iv}-30II''-114I'^2)s_2^8/8!-(6I^{v}-426I'I''-30II'''+$$
$$126I^2I')s_2^9/9!+\text{etc.}$$

So that $S=O$ is the equation of the osculating conic and $I=O$, $I'=O$, are respectively the differential equations of the parabola and general conic respectively.

$$L_1S=-126I's_2^7/7!-168I''s_2^8/8!-(216I'''-1836II')s_2^9/9!-\text{etc.}$$

$$L_2S=36I's_2^6/6!+42I''s_2^7/7!+(48I'''-576II')s_2^8/8!$$

$$+(54I^{iv}-774II''-1782I'^2)s_2^9/9!+\text{etc.}$$

Therefore $$W_1\equiv-5I''L_1S+6I'^2L_1^3-15I'L_2S$$

$$=-(720I'I'''-840I''^2-1080II'^2)s_2^8/8!-(810I'I^{iv}-$$

$$1080I''I'''-2430II'I''+486I'^3)s_2^9/9!+\text{etc.}$$

and $$W_2\equiv7I'(-15S+3I'L_1^2L_2-I''L_1^3)+5(I'''+9II')L_1S$$

$$=(630I'I^{iv}-840I''I'''-1890II'I''+10962I'^3)s_2^8/8!$$

$$+(630I'I^{iv}-1080I'''^2-3690II'I'''+34650I'^2I''+3240I^2I'^2)$$

$$s_2^9/9!+\text{etc.}$$

Hence $$W_1(630I'I^{iv}-840I''I'''-1890II'I''+10962I'^3)$$

$$+W_2(720I'I'''-840I''^2-1080II'^2)=0$$

or, $$7(15I'I^{iv}-20I''I'''-45II'I''+261I'^3)W_1$$

$$+20(6I'I'''-7I''^2-9II'^2)W_2=0$$

is the equation of the osculating cubic.

The differential equation of the general cubic is then

$$21(15I'I^{iv}-20I''I'''-45II'I''+261I'^3)(15I'I^{iv}-20I''I'''$$

$$-45II'I''+9I'^3)$$

$$=100(6I'I'''-7I''^2-9II'^2)\ (7I'I^{v}-12I'''^2-41II'I'''+385I'^2I''$$

$$+36I^2I'^2)$$

The above direct forms and methods of deduction of the equation of the osculating cubic and the general differential equation of the cubic are interesting.

For the Laguerre-Forsyth forms see "Projective Differential Geometry, etc." by Wilczynski.

PARAMETRIC COEFFICIENTS IN THE DIFFERENTIAL GEOMETRY OF CURVES IN AN N-SPACE.[1]

BY

SYAMADAS MUKHOPADHYAYA.

I. GENERAL CONCEPTIONS.

1. Elementary Formulae of n-dimensional Geometry.

The methods explained in the following articles apply to a curve in space of any number of dimensions. A brief exposition, at the outset, of the fundamental conceptions of n-dimensional geometry, and the logical deduction of some of the elementary formulae, may not, therefore, be uninteresting or irrelevant.

By an *n-dimensional space*, or an *n-dimensional*, is here meant an *Euclidean* space of n-dimensions. Through any two points of it can be defined a *straight line*, which lies *entirely* in the given n-dimensional, and through any point of it can be drawn a system of *n axes*, and only n, which are *mutually orthogonal*.

The author believes that every geometry of an n-space, where in every infinitesimal portion Euclidean geometry holds, can be imagined as lying in an Euclidean space of a higher number of dimensions; it may be of an infinite number of dimensions. From this point of view the study of Euclidean space of n-dimensions, leading finally to that of an Euclidean space of an infinite number of dimensions, is of importance. Besides, the investigations in the following papers may, by the use of a suitable distance formula, be adapted to any kind of non-Euclidean space, without insurmountable difficulties.

In an n-dimensional there can be inscribed any number of *lower* dimensionals. Any two points of it determine a *one-dimensional* or straight line, which may be called the *join* of the two points. The

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[1] From Bulletin, Calcutta Mathematical Society, Vol. I, No. 2, 1909.

join will generally mean the *indefinite* straight line through the two points. Finitely interpreted, it will mean the portion of it lying between the two points. If a third point be taken outside this straight line and joined to every point of the straight line, the aggregate of straight lines, thus obtained, is a *two-dimensional* or plane, through the three given points. This plane will be called the *join* of the three given points. The join will generally mean the indefinite plane through the three given points. Finitely interpreted it will mean the portion of the plane bounded by joins of the three points in pairs. By the axiom of Euclidean geometry, only one straight line can be drawn through the third point, in the given n-dimensional, parallel to the join of the first two.

Similarly, if a fourth point be taken, outside the join of the first three points, the aggregate of straight lines from the fourth point to every point of the join of the first three, is a *three-dimensional*, which may be called the *join* of the four given points. This join will generally mean the indefinitely extended three-dimensional, but, finitely interpreted, will mean the portion bounded by the joins of the four given points taken in threes. Through the fourth point there can be only one two-dimensional which is parallel to the join of the first three. This parallel two-dimensional is the aggregate of straight lines through the fourth point, parallel to all straight lines in the join of the first three points.

Generally, an r-dimensional, where r is less than n, is determined as the join of $r+1$ given points, provided no two groups of points, taken from the given points, are *con-joint*, that is, determine the identical dimensional as their joins. Through the $r+1^{th}$ point, only one $(r-1)$-dimensional can be drawn, parallel to the join of the first r points. This $(r-1)$-dimensional is the aggregate of all straight lines, through the $r+1^{th}$ point, which are parallel to all straight lines in the join of the first r points.

Of the n mutually orthogonal axes, through any point of an n-dimensional, any r axes determine an r-dimensional, which may be called the *join* of the r axes. If a plane be drawn through any two axes, all the remaining axes will be perpendicular to this plane, for a line is perpendicular to a plane if it is perpendicular to two straight lines in the plane, which are not parallel. Hence, if any two axes be replaced by two other axes, at right angles, in the same plane, the system of axes will continue to be mutually orthogonal. If the r axes be repeatedly varied in this way, they will have all possible positions in the r-dimensional, which they determine as their join. The join of any r axes is orthogonal to the join of the remaining $n-r$ axes, for any straight line in the former is at right angles to any straight line in the latter. The two mutually orthogonal axial joins have only one point common, which is the origin.

If P be a point outside an r-dimensional, contained in the n-dimensional, then only one perpendicular can be drawn from P to the r-dimensional. For, if two perpendiculars PQ and PR were possible, then the join QR would be in the r-dimensional, and PQR would be a plane triangle having two angles right angles, which is contrary to Euclidean geometry. The foot of the perpendicular from P on the r-dimensional may be called the *projection* of P on the r-dimensional. If any point be taken in the r-dimensional, and from it be drawn the $(n-r)$-dimensional orthogonal to the r-dimensional, then the point of intersection of the parallel $(n-r)$-dimensional through P, with the r-dimensional, is the projection of P on the latter.

If PQ and L be any two straight lines in an n-dimensional, and P', Q' the projections of P, Q on L, then it follows that $P'Q'=PQ\cos\theta$, where θ is the angle between the straight lines. For P' and Q' are the intersections with L, of two parallel $(n-1)$-dimensionals, through P and Q, which are orthogonal to L. If through Q' be drawn $Q'R$, equal and parallel to QP, and PR and $P'R$ be joined, then $PQQ'R$ is a parallelogram. Therefore, PR is parallel to QQ' and consequently lies in the parallel $(n-1)$-dimensional through P. Therefore, also $P'R$ lies in the same $(n-1)$-dimensional and is, consequently, at right angles to $P'Q'$. Hence, $P'Q'=Q'R\cos\theta=PQ\cos\theta$.

Let P be a point in an n-dimensional and $M_1, M_2, M_3 \dots M_n$ its projections on the n axes, mutually orthogonal, through any origin O, in the n-dimensional. Then OM_1, $OM_2 \dots$, OM_2 are the n co-ordinates of P, which may be denoted by $x_1, x_2, \dots x_n$. If a chain of lines $OO_1 O_2 \dots O_n$ be drawn in the n-dimensional from the origin O, and if the successive links of this chain be equal and parallel to $x_1, x_2 \dots x_n$ respectively, O_n must coincide with P. For, the projection of this chain on the r^{th} axes is x_r, and the projection of OP on the same axis is also x_r. Therefore, the projection of PO_n, which is the difference of these two projections, is *nil*, on the r^{th} axis, which may be any of the n axes. Hence PO_n must either vanish or be perpendicular to each of the n axes. But in an n-dimensional there cannot be more than n axes mutually orthogonal. Hence, P must coincide with O_n.

It follows that

$$x_1{}^2+x_2{}^2+\ldots+x_n{}^2=r^2$$

$$l_1{}^2+l_2{}^2+\ldots+l_n{}^2=1$$

where r is equal to OP and $l_1, l_2, \ldots, l_n$ are the direction cosines of OP with reference to the n axes.

Also,

$$rr' \cos\theta=x_1x_1'+x_2x_2'+\ldots+x_nx_n'$$

where $x_1', x_2', \ldots x_n'$ are the co-ordinates of any other point P', r' is equal to OP' and θ is the angle POP'.

Let P, P', P'', P'''' be any four points in an n-dimensional, whose co-ordinates are, respectively,

$$x_1, x_2, \ldots x_n;\ x_1', x_2', \ldots x_n';\ x_1'', x_2'', \ldots x_n'';\ x_1''', x_2''', \ldots x_n'''.$$

Then, by squaring the matrix,

$$\left|\begin{matrix} x_1'-x_1, & x_2'-x_2, & \ldots, & x_n'-x_n \\ x_1''-x_1, & x_2''-x_2, & \ldots, & x_n''-x_n \\ x_1'''-x_1, & x_2'''-x_2, & \ldots, & x_n'''-x_n \end{matrix}\right|$$

is obtained the identity

$$\left|\begin{matrix} \Sigma(x_r'-x_r)^2, & \Sigma(x_r'-x_r)(x_r''-x_r), & \Sigma(x_r'-x_r)(x_r'''-x_r) \\ \Sigma(x_r''-x_r)(x_r'-x_r), & \Sigma(x_r''-x_r)^2, & \Sigma(x_r''-x_r)(x_r'''-x_r) \\ \Sigma(x_r'''-x_r)(x_r'-x_r), & \Sigma(x_r'''-x_r)(x_r''-x_r), & \Sigma(x_r'''-x_r)^2 \end{matrix}\right|$$

$$=\Sigma\left|\begin{matrix} x_{r_1}'-x_{r_1}, & x_{r_2}'-x_{r_2}, & x_{r_3}'-x_{r_3} \\ x_{r_1}''-x_{r_1}, & x_{r_2}''-x_{r_2}, & x_{r_3}'''-x_{r_3} \\ x_{r_1}'''-x_{r_1}, & x_{r_2}'''-x_{r_2}, & x_{r_3}'''-x_{r_3} \end{matrix}\right|^2$$

$$=\Sigma\left|\begin{matrix} 1, & x_{r_1}, & x_{r_2}, & x_{r_3} \\ 1, & x_{r_1}', & x_{r_2}', & x_{r_3}' \\ 1, & x_{r_1}'', & x_{r_2}'', & x_{r_3}'' \\ 1, & x_{r_1}''', & x_{r_2}''', & x_{r_3}''' \end{matrix}\right|^2 \equiv K^2{}_3, \text{ suppose,}$$

where r_1, r_2, r_3 each varies through $1, 2, \ldots, n$.

Therefore,

$$K_3^2 = \begin{vmatrix} 1, \cos P'PP'', \cos P'PP''' \\ \cos P''PP', 1, \cos P''PP''' \\ \cos P'''PP', \cos P'''PP'', 1 \end{vmatrix} PP'^2 . PP''^2 . PP'''^2 .$$

which shows that K_3^2 is an invariant, independent of the origin and orientation of the system of axes. K_r^2 may be similarly defined, where r is not greater than n.

If the system of n axes be so chosen, that the first three lie in the join of P, P', P'', P''', and the rest are all perpendicular to this join, then all the co-ordinates of P, P', P'', P''' vanish, except the first three. Hence in this case,

$$K_3^2 = \begin{vmatrix} 1, x_1, & x_2, & x_3 \\ 1, x_1', & x_2', & x_3' \\ 1, x_1'', & x_2'', & x_3'' \\ 1, x_1''', & x_2''', & x_3''' \end{vmatrix}^2$$

so that K_3 may be taken equal, in magnitude as well as *sign*, to

$$\begin{vmatrix} 1, x_1, & x_2, & x_3 \\ 1, x_1', & x_2', & x_3' \\ 1, x_1'', & x_2'', & x_3'' \\ 1, x_1''', & x_2''', & x_3''' \end{vmatrix}$$

If, further, the first two axes be taken in the join of P, P', P'', then the third co-ordinates of these points also vanish, and the third co-ordinate x_3''' of P''' is the perpendicular h_3 from P''' on the join of P, P', P'', with its proper sign.

Hence,

$$K_3 = \begin{vmatrix} 1, x_1, & x_2, & 0 \\ 1, x_1', & x_2' & 0 \\ 1, x_1'', & x_2'', & 0 \\ 1, x_1''', & x_2''', & x_3''' \end{vmatrix} = x_3''' \begin{vmatrix} 1, & x_1, & x_2 \\ 1, & x_1' & x_2' \\ 1, & x_1'', & x_2'' \end{vmatrix} = K_2 h_3$$

Similarly, the perpendicular h_r to an $(r-1)$-dimensional, determined by r points in an n-dimensional, from an $(r+1)^{th}$ point, is shown to be equal to K_r/K_{r-1}. Evidently, $K_0=1$ and $K_1=PP'$.

Let $P, P', P'', \ldots, P^{(r)}$ be any $r+1$ points in an n-dimensional, U_r the r-dimensional content of their join and U_{r-1} the $(r-1)$-dimensional content of the join of the first r points. If the joins of $P^{(r)}$ with each of the first r points be divided in the same ratio, the join of the points of division will be an $(r-1)$-dimensional V_{r-1}, parallel to U_{r-1}. Let the perpendicular from $P^{(r)}$ on V_{r-1} be kh_r, then the content of the join V_{r-1} will be $k^{r-1}U_{r-1}$.

Therefore, $U_r=\int_{k=0}^{k=1}(k^{r-1}U_{r-1})d(kh_r)=U_{r-1}h_r\int_0^1 k^{r-1}dk=\frac{1}{r}U_{r-1}hr,$

or, $U_r/U_{r-1}=\frac{1}{r}hr=\frac{1}{r}K_r/K_{r-1}$.

Similarly, $U_{r-1}/U_{r-2}=\frac{1}{r-1}K_{r-1}/K_{r-2}$, etc.,

and $U_1=PP'=K_1$, therefore, $U_r=\frac{1}{r!}K_r$[1]

2. Parametric Co-efficients defined in n-dimensional Space.

Let the co-ordinates of a point P on a curve, in n-dimensional space, be $x_1, x_2, \ldots, x_n$, with reference to n axes, mutually orthogonal, through a given origin O. Suppose $x_1, x_2, \ldots, x_n$ are given functions of a parameter t, which are uniform, finite and continuous, within the limits of value considered of the parameter t and are indefinitely derivable.

If D represent the operator $\frac{d}{dt}$, then

$$(m_1, m_2)\equiv\Sigma D^{m_1}x_r\, D^{m_2}x_r,\ r=1, 2, \ldots, n,$$

will be called a *Parametric Coefficient of Class 1.*

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[1] Mr. W. J. C. Sharp, M.A., has called the finite join of $r+1$ points a *simplicissima* of r-dimensions, after Sylvester, and has investigated its properties. (Vide *Proceedings of the London Mathematical Society*, Vols. XVIII and XIX.)

Again, if $[m_1, m^2] \equiv \Sigma \begin{vmatrix} D^{m_1} x_{r_1}, & D^{m_2} x_{r_1} \\ D^{m_1} x_{r_2}, & D^{m_2} x_{r_2} \end{vmatrix}^2$

where $r_1 = 1, 2, \ldots n$, and $r_2 = 1, 2, \ldots, n$, $(r_1 < r_2)$ then $[m_1, m_2]$ will be called a *Parametric Coefficient of Class 2.*

Similarly, if $[m_1, m_2, m_3]^2 \equiv \Sigma \begin{vmatrix} D^{m_1} x_{r_1}, & D^{m_2} x_{r_1}, & D^{m_3} x_{r_1}, \\ D^{m_1} x_{r_2}, & D^{m_2} x_{r_2}, & D^{m_3} x_{r_2}, \\ D^{m_1} x_{r_3}, & D^{m_2} x_{r_3}, & D^{m_3} x_{r_3} \end{vmatrix} 2$

where $r_1 = 1, 2, \ldots, n$; $r_2 = 1, 2, \ldots, n$, $r_3 = 1, 2, \ldots, n$, $(r_1 < r_2 < r_3)$ then $[m_1, m_2, m_3]$ will be called a *Parametric Coefficient of Class 3*, and so on.

Finally, $[m_1, m_2, \ldots m_n] \equiv \begin{vmatrix} D^{m_1} x, \ldots . D^{m_n} x \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ D^{m_1} x \ldots . D^{m_n} x \end{vmatrix}$

will be called a *Parametric Coefficient of Class n*, or highest class.

Further, if,

$$[m_1, m_2 \mid p_1, p_2] = \Sigma \begin{vmatrix} D^{m_1} x_{r_1}, & D^{m_2} x_{r_1} \\ D^{m_1} x_{r_2}, & D^{m_2} x_{r_2} \end{vmatrix} \begin{vmatrix} D^{p_1} x_{r_1}, & D^{p_2} x_{r_1} \\ D^{p_1} x_{r_2}, & D^{p_2} x_{r_2} \end{vmatrix}$$

where $r_1 = 1, 2, \ldots, n$; $r_2 = 1, 2, \ldots, n$, $(r_1 < r_2)$, then $[m_1, m_2 | p_1, p_2]$ will be called a *Parametric Moment of Class 2*, or moment of parametric coefficients $[m_1, m_2]$ and $[p_1, p_2]$.

Similarly, if

$[m_1, m_2, m_3 \mid p_1, p_2, p_3] \equiv$

$$\Sigma \begin{vmatrix} D^{m_1} x_{r_1}, & D^{m_2} x_{r_1}, & D^{m_3} x_{r_1} \\ D^{m_1} x_{r_2}, & D^{m_2} x_{r_2}, & D^{m_3} x_{r_2} \\ D^{m_1} x_{r_3}, & D^{m_2} x_{r_3}, & D^{m_3} x_{r_3} \end{vmatrix} \begin{vmatrix} D^{p_1} x_{r_1}, & D^{p_2} x_{r_1}, & D^{p_3} x_{r_1} \\ D^{p_1} x_{r_2}, & D^{p_2} x_{r_2}, & D^{p_3} x_{r_2} \\ D^{p_1} x_{r_3}, & D^{p_2} x_{r_3}, & D^{p_3} x_{r_3} \end{vmatrix}$$

where $r_1=1, 2, \ldots n$; $r_2=1, 2, \ldots, n$; $r_3=1, 2, \ldots, n$, $(r_1<r_2<r_3)$ then $[m_1, m_2, m_3 \mid p_1, p_2, p_3]$ will be called a *Parametric Moment of Class 3*, or moment of the parametric coefficients $[m_1, m_2, m_3]$ and $[p_1, p_2, p_3]$, and so on.

It is apparent that (m_r, m_r) and (m_r, m_s) may also be written as $[m_r]^2$ and $[m_r \mid m_s]$, respectively.

If the Coefficients be *identical* or of the *highest class*, their moment is equal to their product.

Thus, $$[m_1, m_2 \mid m_1, m_2]=[m_1, m_2]^2$$

and $$[m_1, m_2, \ldots, m_n \mid p_1, p_2 \ldots, p_n]$$
$$=[m_1, m_2, \ldots, m_n][p_1, p_2, \ldots, p_n].$$

These moments are expressible as simple determinant functions of parametric coefficients of Class 1.

Thus, $$[m_1, m_2 \mid p_1, p_2]=\begin{vmatrix}(m_1, p_1), & (m_2, p_1) \\ (m_1, p_2), & (m_2, p_2)\end{vmatrix}$$

$$[m_1, m_2, m_3 \mid p_1, p_2, p_3]=\begin{vmatrix}(m_1, p_1), & (m_2, p_1), & (m_3, p_1) \\ (m_1, p_2), & (m_2, p_2), & (m_3, p_2) \\ (m_1, p_3), & (m_2, p_3), & (m_3, p_3)\end{vmatrix}$$

and so on.

Again, since for space of n-dimensions, parametric coefficients of a class higher than n, must vanish, it follows

$$[m_1, m_2, \ldots m_{n+1}]=0, \; [p_1, p_2, \ldots, p_{n+1}]=0,$$

and, therefore, $$\begin{vmatrix}(m_1, p_1), \ldots, (m_{n+1}, p_1) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (m_1, p_{n+1}) \ldots (m_{n+1}, p_{n+1})\end{vmatrix}=0.$$

If the parameter t be supposed independent of the axes of coordinates, the parametric coefficients will be *invariants*. For, $x_1, x_2, \ldots x_n$ are the components of the position vector of P, and, consequently, their r^{th} derivatives, $D^r x_1, D^r x_2, \ldots, D^r x_n$, are the components of the vectors which represents the r^{th} rate of change of the position-vector of P. It follows that the parametric coefficients of class 1 are invariants, and as the higher parametric coefficients and moments are expressible in terms of parametric coefficients of class I, these also will be invariants.

If through the point P, of the curve, a straight line PT_r be drawn having direction cosines proportional to $D^r x_1, D^r x_2, \ldots, D^r x_n$, then PT_r may be called the r^{th} *variation axis* of the curve at P, with respect to the parameter t. The length of the r^{th} variation axis may be taken equal to $(r, r)^{\frac{1}{2}}$, but when length is not mentioned or implied, the r^{th} variation axis will mean the indefinite straight line in the direction PT_r.

If PT_{m1}, PT_{m2}, etc., be any number r, less than n, of variation axes, and U_r be the content of their join, then, evidently,

$$U_r = \frac{1}{r!}\begin{vmatrix} D^{m_1}x_1, \ldots\ldots, D^{m_1}x \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ D^{m_r}x_r, \ldots\ldots, D^{m_r}x_r \end{vmatrix} = \frac{1}{r!}[m_1, m_2, m_3],$$

provided, the first r co-ordinate axes be taken in the join of the r variation axes, the remaining co-ordinate axes being perpendicular to this join. The above equation fixes the *sign* of $[m_1, m_2, \ldots, m_r]$ and removes an ambiguity of sign, implied in its definition.

Hence, $\qquad [m_1, m_2] = -[m_2, m_1],$

$[m_1, m_2, m_3] = [m_2, m_3, m_1] = -[m_3, m_2, m_1]$, and so on.

3. The n Intrinsic Parameters of a Curve and their Geometrical Interpretations.

The *first intrinsic parameter* of a curve in n-dimensional space may be defined as

$$s_1 = \int_{t_0}^{t} (1, 1)^{\frac{1}{2}} dt$$

The *second intrinsic parameter* may be defined as $s_2 = \int_{t_0}^{t} [1, 2]^{\frac{1}{3}} dt$.

The *third intrinsic parameter* may be defined as $s_3 = \int_{t_0}^{t} [1, 2, 3]^{\frac{1}{6}} dt$

and so on.

The n^{th} *intrinsic parameter* is $s_n=\int_{t_0}^{t}[1,\ 2,\ 3,\dots,\ n]^{\frac{2}{n(n+1)}}dt.$

A plane curve has only two intrinsic parameters s_1 and s_2, and a curve in space only three, s_1, s_2 and s_3.

Let P_0 and P be points on the curve, corresponding to values t_0 and t of the parameter t. Take a large number N, of consecutive points on the curve from P_0 to P, corresponding to equal small increments δt of t, so that $N\delta t=t-t_0$. Then if $(x_1,\ x_2,\dots,x_n)$ and $(x_1+\delta x_1,\ x_2+\delta x_2,\dots,\ x_n+\delta x_n)$ be the co-ordinates of any two consecutive points P_r, P_{r+1}, the length of chord $P_r\ P_{r+1}$ is

$$\{(\delta x_1)^2+(\delta x_2)^2+\dots+(\delta x_n)^2\}^{\frac{1}{2}}$$

and the sum of the lengths of N such chords is

$$\Sigma\{(\delta x_1)^2+(\delta x_2)^2+\dots+(\delta x_n)^2\}^{\frac{1}{2}}=\Sigma\ \left\{\left(\frac{\delta x_1}{\delta t}\right)^2+\left(\frac{\delta x_2}{\delta t}\right)^2 + \left(\frac{\delta x_n}{\delta t}\right)^2\right\}^{\frac{1}{2}}dt.$$

This sum has in general a limiting value, when N is infinitely large, which may be written as $\int_{t_0}^{t}(1,\ 1)^{\frac{1}{2}}dt$, and which is, therefore, the first intrinsic parameter s_1. The first intrinsic parameter is evidently the same as arc-length s.

Again if P_r, P_{r+1}, P_{r+2} be any three consecutive points on the curve, corresponding to equal small increments δt of t, then if x_1, $x_2,\dots,\ x_n$ be the co-ordinates of P_n, those of P_{r+1} and P_{r+2} will be

$$x_1+\delta x_1,\ x_2+\delta x_2,\dots.\ x_n+\delta x_n$$

and $\quad x_1+2\delta x_1+\delta^2 x_1,\ x_2+2\delta x_2+\delta^2 x_2,\dots,\ x_n+2\delta x_n+\delta^2 x_n.$

Hence, the projection of the area δU_2 of the triangle $P_r\ P_{r+1}\ P_{r+2}$

on the plane of the axes x_{r_1}, x_{r_2} is

$$\frac{1}{2!}\begin{vmatrix} 1, & x_{r_1} & x_{r_2} \\ 1, & x_{r_1}+\delta x_{r_1}, & x_{r_2}+\delta x_{r_2} \\ 1, & x_{r_1}+2\delta x_{r_1}+\delta^2 x_{r_1}, & x_{r_2}+2\delta x_{r_2}+\delta^2 x_{r_2} \end{vmatrix}$$

$$=\frac{1}{2!}\begin{vmatrix} \frac{\delta x_{r_1}}{dt}, & \frac{\delta x_{r_2}}{\delta t} \\ \frac{\delta^2 x_{r_1}}{dt^2}, & \frac{\delta^2 x_{r_2}}{\delta t^2} \end{vmatrix}\delta t^3$$

Therefore $$(\delta U_2)^2=\frac{1}{(2!)^2}\Sigma\begin{vmatrix} Dx_{r_1}, & Dx_{r_2} \\ D^2 x_{r_1}, & D^2 x_{r_2} \end{vmatrix}^2 \delta t^6,$$

so that, $$\int_{P_0}^{P}(\delta U_2)^{\frac{1}{3}}=\frac{1}{(2!)^{\frac{1}{3}}}\int_{t_0}^{t}[1,\ 2]^{\frac{1}{3}}\,dt, \text{ or, } s_2=(2!)^{\frac{1}{3}}\int_{P_0}^{P}(\delta U_2)^{\frac{1}{3}}$$

Similarly, $$s_3=(3!)^{\frac{1}{6}}\int_{P_0}^{P}(\delta U_3)^{\frac{1}{6}},$$

where δU_3 is the volume of the join of four consecutive points, and so on.

The r^{th} intrinsic parameter $$s_r=(r!)^{\frac{2}{r(r+1)}}\int_{P_0}^{P}(\delta U_r)^{\frac{2}{r(r+1)}}$$

where δU_r is the content of the join of $r+1$ consecutive points.

It will appear from the above geometrical interpretations that the n quantities $s_1, s_2, s_3, \ldots, s_n$ are intrinsically connected with the curve, and give, as it were, its measure in, respectively, one, two....., n dimensions. The values of $s_1, s_2, s_3, \ldots, s_n$ are independent of the system of axes chosen and of the parameter t and only vary with the positions of P_0 and P on the curve. The n co-ordinates of a point P on the curve may be expressed as functions of any one of these parameters. Any $n-1$ independent equations between these intrinsic parameters will determine a curve in n-dimensional space, intrinsically.

If to any point on an n-dimensional curve, correspond a point on a straight line, such that the *first intrinsic parameter* is the same for both, between pairs of corresponding points, then the straight line is the *rectification* of the curve. Again, if to any point on an n-dimensional curve correspond a point on a plane curve, such that the *first two intrinsic parameters* are the same for both, between pairs of corresponding points, then the latter may be called the *planification* of the curve, or *rectification in two dimensions*. *Rectification in three and higher dimensions* may be similarly defined. In the planification of a three-dimensional curve, each element may be conceived to turn about a consecutive element as an axis, maintaining the angle between them constant. If a model of the curve be constructed of consecutive rigid links ABC, BCD, CDE, etc., slightly bent at B, C, D, etc., and if the common parts BC, CD, etc., be pivoted into each other, the process of planification can be mechanically exhibited.

4. The $n-1$ radii of intrinsic curvature of a curve in n-dimensional space.

If $s_1, s_2, \ldots, s_n$ be the n intrinsic parameters, then

$$\frac{ds_1}{dt}=(1,1)^{\frac{1}{2}},\ \frac{ds_2}{dt}=[1,2]^{\frac{1}{3}},\ \&c.,\ \frac{ds_n}{dt}=[1,2,\ \ ,n]^{\frac{2}{n(n+1)}}$$

The radius of first intrinsic curvature may be defined as

$\rho_1=\left(\frac{ds_1}{ds_2}\right)^{\frac{3}{2}}=\frac{(1,\ 1)^{\frac{3}{2}}}{[1,2)}$, that of second intrinsic curvature, as $\rho_2=$

$\left(\frac{ds_2}{ds_3}\right)^{6}=\frac{[1,2]^2}{[1,2,3]}$, and so on.

The radius of $n-1^{th}$ intrinsic curvature is

$$\rho_{n-1}=\left(\frac{ds_{n-1}}{ds_n}\right)^{\frac{n(n+1)}{2}}=\frac{[1,2,\ldots,n-1]^{\frac{n+1}{n-1}}}{[1,2,\ldots,n]}$$

The dimensions of ρ_1, ρ_2, etc., are obviously of the first degree in length. It is easily shewn that the first radius of intrinsic curvature represents the radius of a circle through three consecutive points.

For, if P, P', P'' be three consecutive points, whose co-ordinates are

$$x_1, x_2, \ldots, x_n;\ x_1+\delta x_1, x_2+\delta x_2, \ldots, x_n+\delta x_n;\ x_1+2\delta x_1, +\delta^2 x_1,$$
$$x_2+2\delta x_2+\delta^2 x_2, \ldots, x_n+2\delta x_n+\delta^2 x_n;$$

then the radius ρ of the circle circumscribing $PP'P''$ is $PP' \cdot P'P'' \cdot P''P/4U_2$ where U_2 is the area of triangle $PP'P''$.

But $PP'=\{\Sigma(\delta x)^2\}^{\frac{1}{2}}=(1, 1)^{\frac{1}{2}}\delta t$, ultimately, $P'P''=PP'$, $PP''=2PP'$ ultimately, and $4U_2=2[1, 2]\delta t^3$, ultimately.

Therefore, $$\rho=\frac{(1, 1)^{\frac{3}{2}}}{[1, 2]}=\rho_1.$$

The higher intrinsic parameters are expressible in terms of the first intrinsic parameter and the radii of intrinsic curvature.

For, $$(1, 1)^{\frac{1}{2}}=\frac{ds_1}{dt}=\frac{ds}{dt};\ [1, 2]^{\frac{1}{3}}=\rho_1^{-\frac{1}{3}}\ (1, 1)^{\frac{1}{2}}=\rho_1^{-\frac{1}{3}}\ \frac{ds}{dt};$$

$$[1, 2, 3]^{\frac{1}{6}}=\rho_2^{-\frac{1}{6}}\ [1, 2]^{\frac{1}{3}}=\rho_1^{-\frac{1}{3}}\ \rho_2^{-\frac{1}{6}}\frac{ds}{dt}.$$

Therefore, $$s_2=\int_{t_0}^{t} [1, 2]^{\frac{1}{3}}dt=\int_{P_0}^{P} \rho_1^{-\frac{1}{3}}\ ds$$

Similarly, $$s_3=\int_{t_0}^{t} [1, 2, 3]^{\frac{1}{6}}dt=\int_{P_0}^{P} \rho_1^{-\frac{1}{3}}\rho_2^{-\frac{1}{6}}ds$$

and $$s_n=\int_{t_0}^{t} [1, 2, \ldots, n]^{\frac{2}{n(n+1)}}dt=\int_{s_0}^{s} \rho_1^{-\frac{1}{3}}\rho_2^{-\frac{1}{6}}\ldots\rho_{n-1}^{-\frac{2}{n(n+1)}}ds.$$

The invariance of s_2, $s_3, \ldots, s_n$ is, therefore, connected with the invariance of ρ_1, $\rho_2 \ldots \rho_{n-1}$ and s.

5. The n principal axes of a curve, in n-dimensional space.

If through any point P of a curve, in n-dimensional space, be drawn the first r variation axes, where r is less than n, then the r-dimensional determined as their join may be called the *r-dimensional extension* of the curve at P. Thus the first variation axis or tangent is the *rectilinear extension* of the curve and the osculating plane, which is the join of the first two variation axes, is the *plane extension*.

Let x_1, x_2,...., x_n be the co-ordinates of any point Q in the r-dimensional extension of the curve at P. Let a_1,...., a_r be the co-ordinates of Q, referred to the first r variation axes at P. Then

$$X_1-x_1=a_1Dx_1/(1,\ 1)^{\frac{1}{2}}+a_2D^2x_1/(2,\ 2)^{\frac{1}{2}}+\ldots+a_rD^rx_1/(r,\ r)^{\frac{1}{2}}$$
$$=C_1Dx_1+C_2D^2x_1+\ldots+C_rD^rx_1$$

where C_1, C_2,...., C_r are arbitrary constants.

Similarly,
$$X_2-x_2=C_1Dx_2+C_2D^2x_2+\ldots+C_rD^rx_2$$

....

$$X_n-x_n=C_1Dx_n+C_2D^2x_n+\ldots+C_rD^rx_n$$

Hence, the equations of the r-dimensional extension are

$$\left\| \begin{array}{ccc} X_1-x_1, & X_2-x_2, \;.\;.\;. & X_n-x_n \\ Dx_1, & Dx_2, \;.\;.\;. & Dx_n \\ D^2x_1, & D^2x_2, \;.\;.\;. & D^2x_n \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ D^rx_1, & D^rx_2, \;.\;.\;. & D^rx_n \end{array} \right\| =0.$$

It is easily seen that the r-dimensional extension is the join of $r+1$ consecutive points on the curve. For, if $(x_1, x_2,..., x_n)$, $(x_1', x_2', ..., x'_n)$,..., $(x_1{(r)}, x_2{(r)},...., x_n{(r)})$, be the co-ordinates of any $r+1$ points and X_1, X_2,..., X_n be the co-ordinates of any point in their join, then

$X_1=x_1+A_1(x_1'-x_1)+A_2(x_1''-2x_1'+x_1)+\ldots+A_r(x_1{(r)}-rx_1{(r-1)}$ + &c.), because, on the right-hand side, the sum of the coefficients of x_1, x_1',..., $x_1{(r)}$ is, obviously, unity, and they involve r arbitrary constants A_1, A_2,..., A_r.

If the $r+1$ points be *consecutive*, then the above equation becomes

$$X_1 = x_1 + A_1\delta x_1 + A_2\delta^2 x_1 + .. + A_r\delta^r x_1$$

$$= x_1 + C_1 Dx_1 + C_2 D^2 x_1 + .. + C_r D^r x_1$$

where $C_1 = Lt A_1 \delta t,\ C_2 = Lt A_2 \delta t^2, \ldots,\ C_r = Lt A_r \delta^r t.$

Similarly, $X_2 = x_2 + C_1 Dx_2 + C_2 D^2 x_2 + .. + C_r D^r x_2$

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$$X_n = x_n + C_1 Dx_n + C_2 D^2 x_n + .. + C_r D^r x_n$$

Hence, the r-dimensional extension embraces $r+1$ consecutive points and has, therefore, contact of the r^{th} order with the curve.

Let $A_{1r}, A_{2r}, \ldots, A_{rr}$, where r does not exceed n, be the first minors of $(1, r), (2, r), \ldots (r, r)$, in the determinant,

$$\begin{vmatrix} (1, 1), (1, 2), \ldots, (1, r) \\ (2, 1), (2, 2), \ldots, (2, r) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ (r, 1), (r, 2), \ldots, (r, r) \end{vmatrix} \equiv \triangle^r = [1, 2, .., r]^2,$$

then the straight line PO_r, whose direction cosines are proportional to $A_{1r}Dx_p + A_{2r}D^2x_p + .. + A_{rr}D^r x_p$, where $p=1, 2, \ldots, n$ may be called the r^{th} *principal axis* of the curve at P.

Let the direction cosines of the r^{th} principal axis be

$(A_{1r}Dx_p + A_{2r}D^2x_p + .. + A_{rr}D_r x_p)/\lambda_r$, where $p=1, 2, \ldots, n$.

Then, $\lambda_r{}^2 = \Sigma(A_{1r}Dx_p + A_{2r}D^2x_p + .. + A_{rr}D^r x_p)^2,\ p=1, 2, \ldots, n$

$$= \Sigma A_{qr}\{A_{1r}(1,q) + A_{2r}(2, q) + .. + A_{rr}(r, q)\}, q=1,2..,r$$

$$= A_{rr}\{A_{1r}(1, r) + A_{2r}\ (2, r) + .. + A_{rr}\ (r, r)\}$$

$$= \triangle_{r-1}\triangle_r = [1, 2.., r-1]^2[1, 2,..,r]^2,$$ if r be greater than 2. Therefore, λ_r may be taken equal to

$$[1, 2,.. r-1][1, 2, \ldots, r],$$ if r be greater than 2.

When $r=1$, $\lambda_r = (1, 1)^{\frac{1}{2}} = [1]$ and when $r=2$, $\lambda_r = (1, 1)^{\frac{1}{2}}\ [1, 2] = [1][1, 2]$.

If $[1, 2, \ldots, r-1, x_p | 1, 2, \ldots, r]$ stand for

$$\begin{vmatrix} (1, 1), & (1, 2), \ldots\ldots\ldots\ldots & (1, r) \\ (2, 1), & (2, 2), \ldots\ldots\ldots\ldots & (2, r) \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ (r-1, 1), & (r-1, 2), \ldots, & (r-1, r) \\ Dx_p, & D^2x_p, \ldots, & D^r x_p \end{vmatrix} = A_{1r} Dx_p + A_{2r} D^2 x_p + \ldots + A_{rr} D_r x_p,$$

then the direction cosines of the r^{th} principal axis may be more compactly written as

$$\frac{[1, 2, \ldots, r-1, x_p | 1, 2, \ldots, r]}{[1, 2, \ldots, r-1][1, 2, \ldots, r]}, p=1, 2, \ldots, n.$$

Any two principal axes PO_r and PO_s, where r is supposed greater than s, are mutually orthogonal. For,

$$\Sigma(A_{1r} Dx_p + A_{2r} D^2 x_p + \ldots + A_{rr} D^r x_p)(A_{1s} Dx_p + A_{2s} D^2 x_p + \ldots +$$

$$A_{ss} D^s x_p)$$

$$= \Sigma A_{qs} \{A_{1r}(1, q) + A_{2r}(2, q) + \ldots + A_{rr}(r, q)\}, \qquad q=1, 2 \ldots, s$$

$$= \Sigma A_{qs}\, 0 = 0.$$

It is evident, from definition, that the r^{th} principal axis lies in the r-dimensional extension of the curve and, consequently, in all the higher extensions.

The first principal axis is the tangent to the curve, and the remaining axes are the $n-1$ *principal normals* to the curve.

Let $\xi_1, \xi_2, \ldots, \xi_n$ be the n co-ordinates of a point Q on the curve, referred to the n principal axes of the curve at P. Then, if t be measured from the origin P,

$$(1, 1)^{\frac{1}{2}}\, \xi_1 = (X_1 - x_1) Dx_1 + (X_2 - x_2) Dx_2 + \ldots + (X_n - x_n) Dx_n$$

$$= (1, 1)t + (1, 2)\, t^2/2! + (1, 3)\, t^3/3! + \&c.$$

$$(1, 1)^{\frac{1}{2}}\, [1, 2] \xi_2 = \Sigma(X_p - x_p) \begin{vmatrix} (1, 1), & (1, 2) \\ Dx_p, & D^2 x_p \end{vmatrix}, \; p=1, 2, \ldots, n$$

$$= \begin{vmatrix} (1, 1), & (1, 2) \\ (1, 2), & (2, 2) \end{vmatrix} t^2/2! + \begin{vmatrix} (1, 1), & (1, 2) \\ (1, 3), & (2, 3) \end{vmatrix} t^3/3! + \&c.$$

$$= [1, 2]^2 t^2/2! + [1, 2 | 1, 3] t^3/3! + [1, 2 | 1, 4] t^4/4! + \&c.,$$

Similarly,

$$[1, 2][1, 2, 3]\xi_3 = [1, 2, 3]^2 t^3/3! + [1, 2, 3|1, 2, 4]t^4/4! + \&c.$$

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$$[1, 2, \ldots, r-1][1, 2, \ldots, r]\xi_r = [1, 2, \ldots, r]^2 t^r/r!$$
$$+ [1, 2, \ldots, r|1, 2, \ldots, r-1, r+1]t^{r+1}/(r+1)! + \&c.$$

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$$[1, 2, \ldots, n-1]\xi_n = [1, 2, \ldots, n]t^n/n! + [1, 2, \ldots, n-1, n+1]$$
$$t^{n+1}/(n+1)! + \&c.$$

It is evident that, when t is an infinitesimal,

$$Lt\frac{\xi_1^2}{\xi_2} = \rho_1 . 2!,\ Lt\frac{\xi_2^3}{\xi_3^2} = \frac{\rho_2^2}{\rho_1} \cdot \frac{3!^2}{2!^3}, \ldots,\ Lt\frac{\xi_{n-1}^n}{\xi_n^{n-1}} = \frac{\rho_{n-1}^{n-1}}{\rho_{n-2}^{n-2}} \frac{n!^{n-1}}{(n-1)!^n}$$

It will be noticed that ξ_2 is of the same sign as ρ_1, hence, the centre of curvature is a point on the second principal axis, at a distance ρ_1 from P. The co-ordinates of the centre of curvature are, therefore,

$$X_p = x_p + \begin{vmatrix} (1, 1), & (1, 2) \\ Dx_p, & D^2x_p \end{vmatrix} \cdot \frac{\rho_1}{(1, 1)^{\frac{1}{2}}[1, 2]} = x_p + \begin{vmatrix} (1, 1), & (1, 2) \\ Dx_p, & D^2x_p \end{vmatrix} \cdot \frac{(1, 1)}{[1, 2]^2}$$

$$= x_p + \frac{[1, x_p|1, 2]\ (1,1)}{[1, 2]^2},\ p = 1, 2, \ldots, n.$$

6. The $n-1$ principal curvatures of a curve in n-dimensional space.

Let $\xi_1, \xi_2, \ldots, \xi_n$ be the n co-ordinates of a point on a curve referred to the n principal axes at P, and let $u_1, u_2, u_3, \ldots, u_n$ be defined by

$$u_1 \equiv Lt\frac{D\xi_2}{\xi_1 D\xi_1} = \frac{[1, 2]}{[1]^3}, u_2 \equiv Lt\frac{D\xi_3}{\xi_2 D\xi_1} \equiv \frac{[1, 2, 3]}{[1, 2]^2}$$

$$u_3 \quad = Lt \frac{D\xi_4}{\xi_3 D\xi_1} = \frac{[1, 2, 3, 4][1, 2]}{[1, 2, 3]^2[1]}, \ldots\ldots,$$

$$u_{n-1} = Lt \frac{D\xi_n}{\xi_{n-1} D\xi_1} = \frac{[1, 2, \ldots, n][1, 2, \ldots, n-2]}{[1, 2, 3, \ldots, n-1]^2[1]},$$

then u_1, u_2, u_3[*], ….., u_{n-1} may be called the $n-1$ *principal curvatures* of the curve at P. Evidently $u_1 = \frac{1}{\rho_1}$ and $u_2 = \frac{1}{\rho_2}$, where ρ_1 and ρ_2 are the first two radii of intrinsic curvature.

If the independent variable be s, then $(1, 1) = 1$, $(1, 2) = \frac{1}{2} D\,(1, 1) = 0$, $[1, 2] = u_1$, $[1, 2 | 1, 3] = \frac{1}{2} D[1, 2]^2$, $[1, 2, 3] = u_1{}^2 u_2$, $[1, 2, 3 | 1, 2, 4] = \frac{1}{2} D[1, 2, 3]^2$, $[1, 2, 3, 4] = u_1{}^3 u_3{}^2 u_3$, $[1, 2, 3, 4 | 1, 2, 3, 5] = \frac{1}{2} D\,[1, 2, 3, 4]^2, \ldots\ldots$, $[1, 2, \ldots, n] = u_1{}^{n-1} u_2{}^{n-2} \ldots u_{n-1}$, $[1, 2, \ldots n-1, n+1] = D\,[1, 2, \ldots n]$.

Thus the first two coefficients of each of the series for ξ_1, ξ_2, …, ξ_n, in ascending powers of s, are at once obtained. General methods will be attempted hereafter, by which each of the above series may be extended to any number of terms.

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[*] The third curvature has been investigated by Brunel, Pirondini, Piccioli, Hardy, Lovett and others.

PARAMETRIC COEFFICIENTS IN THE DIFFERENTIAL GEOMETRY OF CURVES IN AN N-SPACE

BY

SYAMADAS MUKHOPADHYAY.

II. EXTENSION OF SERRET-FRENET FORMULÆ TO CURVES IN n-DIMENSIONAL SPACE.*

1. Notational Recapitulation.

It will be useful at the outset to recapitulate and extend the notational formulæ which have been explained in Paper I.

Let P be any function of a parameter t, continuous and possessing differential coefficients of all orders, within a given domain; then the following notational identities are defined to hold:

$$\left(\frac{d}{dt}\right)^r P \equiv D^r P \equiv (r, P) \equiv (P, r) \equiv [r \mid P] \equiv [P \mid r].$$

Again, if P, Q, R, etc., be any number of functions like P, then the following notational identities are defined to hold:

$$\begin{vmatrix} D^p P, & D^p Q \\ D^q P, & D^q Q \end{vmatrix} \equiv \begin{vmatrix} (p, P), & (p, Q) \\ (q, P), & (q, Q) \end{vmatrix} \equiv [p, q \mid P, Q] \equiv [P, Q \mid p, q]$$

$$\begin{vmatrix} D^p P, & D^p Q, & D^p R \\ D^q P, & D^q Q, & D^q R \\ D^r P, & D^r Q, & D^r R \end{vmatrix} \equiv \begin{vmatrix} (p, P), & (p, Q), & (p, R) \\ (q, P), & (q, Q), & (q, R) \\ (r, P), & (r, Q), & (r, R) \end{vmatrix} \equiv [p, q, r, \mid P, Q, R]$$

$$\equiv [P, Q, R \mid p, q, r], \text{ etc.}$$

If x_1, x_2,.... x_n be the n co-ordinates of a point on a curve in an Euclidean n-dimensional space, referred to orthogonal axes through any origin, then x_1, x_2,...., x_n may be conceived to be given functions of a parameter t. A number of functions of the differential coefficients of x_1, x_2,...., x_n, with respect to t, have been called

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* From *Bulletin, Calcutta Mathematical Society*, Vol. I, 1909.

parametric coefficients and *parametric moments* in Paper I. The distinction between parametric coefficients and parametric moments will be dropped in this paper. They will be called *parametric coefficients* always. They are *invariants* for any *orthogonal* transformations. They are obtained and represented in the following manner:

The expression $\Sigma(p, x_r)(q, x_r)$, where $r=1, 2, \ldots, n$, and p, q are positive integers, is a *parametric coefficient of class I.* If, for brevity, the suffix to x be omitted, then the following identities are *defined* to hold:

$$\Sigma(p, x)(q, x) \equiv (p, q) \equiv (q, p) \equiv [p \mid q] \equiv [q \mid p].$$

The expression (p, p) or $[p \mid p]$ is also written as $[p]^2$, where $[p]$ is the *positive* square root of (p, p).

The expression $\Sigma[p_1, p_2 \mid x_{r_1}, x_{r_2}][q_1, q_2 \mid x_{r_1}, x_{r_2}]$, where $r_1 = 1, 2, 3, \ldots, n$; $r_2 = r_1+1, r_1+2, \ldots, n$ and p_1, p_2, q_1, q_2 are positive integers, is a *parametric coefficient of class II.* If, for brevity, the suffixes to x be omitted, then the following notational identities are defined to hold:

$$\Sigma[p_1, p_2 \mid x, x][q_1, q_2 \mid x, x] \equiv [p_1, p_2 \mid q_1, q_2] \equiv [q_1, q_2 \mid p_1, p_2]$$

The expression $[p_1, p_2 \mid p_1, p_2]$ is also written as $[p_1, p_2]^2$, so that $[p_1, p_2] = \pm\sqrt{[p_1, p_2 \mid p_1, p_2]}$. The method of choosing the proper sign before the radical has been explained in Paper I, whence it follows $[p_2, p_1] = -[p_1, p_2]$.

Similarly, $\Sigma[p_1, p_2, p_3 \mid x_{r_1}, x_{r_2}, x_{r_3}][q_1, q_2, q_3 \mid x_{r_1}, x_{r_2}, x_{r_3}]$, where $r_1 = 1, 2, \ldots, n$; $r_2 = r_1+1, r_1+2, \ldots, n$; $r_3 = r_2+1, r_2+2, \ldots, n$ and p_1, p_2, p_3, q_1, q_2, q_3 are positive integers, is a *parametric coefficient of class III.* If, for brevity, suffixes to x, in the summation, be omitted, then the following notational identities are defined to hold:

$$\Sigma[p_1, p_2, p_3 \mid x, x, x][q_1, q_2, q_3 \mid x, x, x] \equiv [p_1, p_2, p_3 \mid q_1, q_2, q_3] \equiv [q_1, q_2, q_3 \mid p_1, p_2, p_3].$$

The expression $[p_1, p_2, p_3 \mid p_1, p_2, p_3]$ is also written as $[p_1, p_2, p_3]^2$, so that $[p_1, p_2, p_3] = \pm\sqrt{[p_1, p_2, p_3, \mid p_1, p_2, p_3]}$. The method of choosing the proper sign before the radical has been explained in Paper I, whence it follows that

$$[p_1, p_2, p_3] = -[p_2, p_1, p_3] = [p_2, p_3, p_1] = -[p_3, p_2, p_1].$$

Higher parametric coefficients are derived and represented on exactly the same principles.

Finally, $[p_1, p_2, \dots, p_n] \equiv [p_1, p_2, \dots, p_n \mid x_1, x_2, \dots x_n]$ is the parametric coefficient of the class n, or the highest class for the given dimension of space.

It follows from theory of matrices that

$$[p_1, p_2, \mid q_1, q_2] = \Sigma [p_1, p_2 \mid x_r, x_{r2}]\,[q_1, q_2 \mid x_{r1}, x_{r2}]$$

$$= \begin{vmatrix} (p_1, x_1), (p_1, x_2), \dots, (p_1, x_n) \\ (p_2, x_1), (p_2, x_2), \dots, (p_2, x_n) \end{vmatrix} \begin{vmatrix} (q_1, x_1), (q_1, x_2), \dots, (q_1, x_n) \\ (q_2, x_1), (q_2, x_2), \dots, (q_2, x_n) \end{vmatrix}$$

$$= \begin{vmatrix} (p_1, q_1), (p_1, q_2) \\ (p_2, q_1), (p_2, q_2) \end{vmatrix}$$

Similarly, $$[p_1, p_2, p_3 \mid q_1, q_2, q_3] = \begin{vmatrix} (p_1, q_1), (p_1, q_2), (p_1, q_3) \\ (p_2, q_1), (p_2, q_2), (p_2, q_3) \\ (p_3, q_1), (p_3, q_2), (p_3, q_3) \end{vmatrix}$$

and so on for higher parametric coefficients.

In addition to the parametric coefficients, explained above, which may be called *pure*, it is useful to consider a class of analogous summational expressions, containing numerical indices p, q, etc., as well as arbitrary functions P, Q, etc. These may be called *mixed* parametric coefficients. They are represented on analogous principles.

For example, the expression $\Sigma[p_1, p_2 \mid x_r, Q][q_1 \mid x_r]$, $r=1, 2, \dots, n$, where Q is any given function of the parameter t, is a mixed parametric coefficient. If, for brevity, the suffix of x be omitted, then the following notational identity may be defined to hold:

$$\Sigma[p_1, p_2 \mid x, Q][q_1 \mid x] \equiv [p_1, p_2 \mid q_1, Q]$$

It follows, at once, that $$[p_1, p_2 \mid q_1, Q] = \Sigma \begin{vmatrix} (p_1, x), (p_1, Q) \\ (p_2, x), (p_2, Q) \end{vmatrix} (q_1, x)$$

$$= \begin{vmatrix} (p_1, x_1), (p_1, x_2), \dots, (p_1, x_n), (p_1, Q) \\ (p_2, x_1), (p_2, x_2), \dots, (p_2, x_n), (p_2, Q) \end{vmatrix}$$

$$\begin{vmatrix} (q_1, x_1), (q_1, x_2), \dots, (q_1, x_n), 0 \\ 0\ , \quad 0\ \dots, \quad 0\ , 1 \end{vmatrix}$$

$$= \begin{vmatrix} (p_1, q_1), (p_1, Q) \\ (p_2, q_1), (p_2, Q) \end{vmatrix}$$

Similarly, $[p_1, p_2, p_3 \mid q_1, q_2, Q]$ may be defined to represent

$$\Sigma[p_1, p_2, p_3 \mid x_{r1}, x_{r2}, Q][q_1, q_2 \mid x_{r1}, x_{r2}],$$

where $r_1=1, 2, \ldots, n$; $r_2=r_1+1, r_1+2, \ldots, n$;

or, omitting suffixes to x,

$$[p_1, p_2, p_3 \mid q_1, q_2, Q] \equiv \Sigma[p_1, p_2, p_3 \mid x, x, Q][q_1, q_2 \mid x, x].$$

But $\Sigma[p_1, p_2, p_3 \mid x, x, Q][q_1, q_2 \mid x, x]$

$$=\Sigma \begin{vmatrix} (p_1, x_{r1}) & (p_1, x_{r2}), & (p_1, Q) \\ (p_2, x_{r1}), & (p_2, x_{r2}), & (p_2, Q) \\ (p_3, x_{r1}), & (p_3, x_{r2}), & (p_3, Q) \end{vmatrix} \begin{vmatrix} (q_1, x_{r1}), & (q_1, x_{r2}) \\ (q_2, x_{r1}), & (q_2, x_{r2}) \end{vmatrix}$$

$$= \begin{vmatrix} (p_1, x_1), & (p_1, x_2), \ldots, & (p_1, x_n), & (p_1, Q) \\ (p_2, x_1), & (p_2, x_2), \ldots, & (p_2, x_n), & (p_2, Q) \\ (p_3, x_1), & (p_3, x_2), \ldots, & (p_3, x_n), & (p_3, Q) \end{vmatrix} \begin{vmatrix} (q_1, x_1), & (q_1, x_2), \ldots, & (q_1, x_n), & 0 \\ (q_2, x_1), & (q_2, x_2) \ldots, & (q_2, x_n), & 0 \\ 0 \;, & 0 \;\; \ldots & 0 & , 1 \end{vmatrix}$$

$$= \begin{vmatrix} (p_1, q_1), & (p_1, q_2), & (p_1, Q) \\ (p_2, q_1), & (p_2, q_2), & (p_2, Q) \\ (p_3, q_1), & (p_3, q_2), & (p_3, Q) \end{vmatrix}$$

Therefore, $[p_1, p_2, p_3, \mid q_1, q_2, Q] = \begin{vmatrix} (p_1, q_1), & (p_1, q_2), & (p_1, Q) \\ (p_2, q_1), & (p_2, q_2), & (p_2, Q) \\ (p_3, q_1), & (p_3, q_2), & (p_3, Q) \end{vmatrix}$

It may be noticed that the above equation is of the same form as for the corresponding pure parametric coefficient.

Again, $[p_1, p_2, P \mid q_1, q_2, Q]$ may be defined to represent $\Sigma[p_1, p_2 \mid x_r, Q][q_1, q_2 \mid x_r, P]$, where $r=1, 2, \ldots, n$, or, omitting suffix to x,

$$[p_1, p_2, P \mid q_1, q_2, Q] = \Sigma[p_1, p_2 \mid x, Q][q_1, q_2 \mid x, P].$$

Similarly, $[p_1, p_2, p_3, P \mid q_1, q_2, q_3, Q] \equiv \Sigma[p_1, p_2, p_3 \mid x, x, Q][q_1, q_2, q_3 \mid x, x, P]$ and $[p_1, p_2, p_3, P \mid q_1, q_2, Q_1, Q_2] = \Sigma[p_1, p_2, p_3 \mid x, Q_1, Q_2][q_1, q_2 \mid x, P]$.

Further, $[p_1, p_2, P]^2$ and $[p_1, p_2, p_3\ P]^2$ represent $[p_1, p_2, P \mid p_1, p_2, P]$ and $[p_1, p_2, p_3, P \mid p_1, p_2, p_3, P]$.

It follows that $[p_1, p_2, P \mid q_1, q_2, Q] = -\begin{vmatrix} (p_1, q_1), & (p_1, q_2), & (p_1, Q) \\ (p_2, q_1), & (p_2, q_2), & (p_2, Q) \\ (P, q_1), & (P, q_2), & 0 \end{vmatrix}$

for, $\Sigma[p_1, p_2 \mid x, Q][q_1, q_2 \mid x, P]$

$$= -\begin{vmatrix} (p_1, x_1), & (p_1, x_2), \ldots, & (p_1, x_n), & (p_1, Q), & 0 \\ (p_2, x_1), & (p_2, x_2), \ldots, & (p_2, x_n), & (p_2, Q), & 0 \\ 0, & 0 \quad \ldots, & 0, & 0, & 1 \end{vmatrix}$$

$$\times \begin{vmatrix} (q_1, x_1), & (q_1, x_2), \ldots, & (q_1, x_n), & 0, & (q_1, P) \\ (q_2, x_1), & (q_2, x_2), \ldots, & (q_2, x_n), & 0, & (q_2, P) \\ 0, & 0 \quad \ldots, & 0, & 1, & 0 \end{vmatrix}$$

Similar determinant expressions are easily obtained for all mixed parametric coefficients.

It may be observed that each mixed parametric coefficient contains a number of *numerical indices*, p_1, p_2, etc., and a number of *functionals* P_1, P_2, etc.; the number of indices exceeds the number of functionals and the indices precede the functionals in the order of representation.

A few general remarks may be made on both kinds of parametric coefficients. Each coefficient consists of an *even* number of elements consisting of numerical indices or numerical indices and functionals. The elements are divided into two groups of equal number, separated by a vertical line and enclosed between a pair of brackets. The left-hand and right-hand groups are interchangeable without change of value of the parametric coefficient. An interchange between two elements both numerical or both functional belonging to the same group produces a change of sign in the value of the parametric coefficient. Hence, if any two elements of the same group be identical, the parametric coefficient vanishes. It has been supposed that the indices are positive integers and the number of indices is greater than the number of functionals. It may be convenient hereafter to introduce and interpret the index zero.

2. Differentiation of Parametric Coefficients.

The parametric expressions, explained in the last article, readily admit of differentiation. Evidently

$$D(r,P)=DD^rP=D^{r+1}P=(r+1,P)$$

$$D[p,q \mid P,Q]=[p+1,q \mid P,Q]+[p,q+1 \mid P,Q]$$

$$D[p,q,r \mid P,Q,R]=[p+1,q,r \mid P,Q,R]+[p,q+1,r \mid P,Q,R]$$
$$+[p,q,r+1 \mid P,Q,R]$$

$$D(p,q)=D\Sigma(p,x_r)(q,x_r)=\Sigma D\{(p,x_r)(q,x_r)\}$$
$$=\Sigma\{(p+1,x_r)(q,x_r)+(p,x_r)(q+1,x_r)\}=(p+1,q)+(p,q+1).$$

Similarly, $D[p_1,p_2 \mid q_1,q_2]=D\Sigma[p_1,p_2 \mid x,x][q_1,q_2 \mid x,x]$

$$=[p_1+1,p_2 \mid q_1,q_2]+[p_1,p_2+1 \mid q_1,q_2]$$
$$+[p_1,p_2 \mid q_1+1,q_2]+[p_1,p_2 \mid q_1,q_2+1]$$

and $D[p_1,p_2,p_3 \mid q_1,q_2,q_3]=[p_1+1,p_2p_3 \mid q_1,q_2,q_3]$

$$+[p_1,p_2+1,p_3 \mid q_1,q_2,q_3]+[p_1,p_2,p_3+1 \mid q_1,q_2q_3]$$
$$+[p_1,p_2p_3 \mid q_1+1,q_2,q_3]+[p_1,p_2,p_3 \mid q_1,q_2+1,q_3]$$
$$+[p_1,p_2,p_3, \mid q_1,q_2,q_3+1].$$

Again, $D[p_1,p_2 \mid q,Q]=D\Sigma[p_1,p_2 \mid x,Q][q \mid x]=\Sigma\{[p_1+1,p_2,x,$ $Q][q \mid x]+[p_1,p_2+1 \mid x,Q][q \mid x]+[p_1,p_2, \mid x,Q][q+1 \mid x]\}$

$$=[p_1+1,p_2 \mid q,Q]+[p_1,p_2+1 \mid q,Q]+[p_1,p_2, \mid q+1,Q].$$

Similarly, $D[p_1,p_2,p_3 \mid q_1,q_2,Q]=[p_1+1,p_2,p_3 \mid q_1,q_2,Q]$

$$+[p_1,p_2+1,p_3 \mid q_1,q_2,Q]+[p_1,p_2,p_3+1 \mid q_1,q_2,Q]$$
$$+[p_1,p_2,p_3 \mid q_1+1,q_2,Q]+[p_1,p_2,p_3 \mid q_1,q_2+1,Q].$$

It will be obvious from the above illustrations, that the simple rule for the differentiation of a parametric coefficient is:

Increase all the *numerical indices*, in the parametric coefficient, one after another, by unity, and take the sum of the expressions thus obtained, the functionals, if any, being left untouched.

It should be pointed out, however, that this rule does not apply directly to the abbreviated forms $[p]$, $[p,q]$, etc. These may be differentiated as follows:

$$\mathrm{D}[p]^2=D(p,p)=(p+1,p)+(p,p+1)$$

or, $2[p]D[p]=2(p,p+1)$.

Therefore $D[p]=\dfrac{(p,p+1)}{[p]}$

Similarly, $D[p,q]=\dfrac{[p+1,\ q \mid p,\ q]+[p,\ q+1 \mid p,\ q\ ]}{[p,\ q]}$, etc.

3. Principal Axes and Principal Curvatures.

In Paper I, the line through the point P on the curve, whose direction cosines are

$$D^r x_1/[r],\ D^r x_2/[r], \ldots,\ D^r x_n/[r],$$

has been called the *rth variation axis* at P. Let the rth variation axis be called T_r. The r-dimensional space which passes through the first r variation axes at P, has been called the r-dimensional *extension* of the curve at P. If this r-dimensional extension be called E_r, then E_r osculates the curve at P. A line through P lying in E_r and normal to the first $r-1$ variation axes has been called the *rth principal axis* of the curve at P. If this rth principal axis be called λ_r, then λ_r is evidently normal to E_{r-1}, $E_{r-2}, \ldots,$ E_r. The direction cosines of λ_r have been shewn equal to

$$\frac{[1,\ 2, \ldots, r \mid 1,\ 2 \ldots, r-1, x_p]}{[1,\ 2, \ldots, r]\,[1, 2, \ldots, r-1]}, p=1,\ 2, \ldots, n.$$

The above direction cosines also define the *sense* of λ_r.

It may be interesting to re-establish this result here, as an exercise in the methods of this paper.

For clearness of reasoning let a particular principal axis be considered, suppose the *fourth*. The proof will equally apply to any other principal axis. It is to be established here that the direction cosines of λ_4 are

$$\frac{[1,\ 2,\ 3, 4 \mid 1,\ 2,\ 3,\ x_p]}{[1,\ 2,\ 3,\ 4][1,\ 2,\ 3,]},\ p=1,\ 2,\ \ldots,\ n.$$

First, the line defined by the above direction cosines is to be shewn to lie in E_4. This follows from the form of $[1,\ 2,\ 3,\ 4 \mid 1\ \ 2\ 3,\ x_p\ ]$ which is *linear* in $(1,\ x_p)$, $(2,\ x_p)$, $(3,\ x_p)$ and $(4,\ x_p)$, for

$$[1,\ 2,\ 3,\ 4 \mid 1,\ 2,\ 3, x_p]=\begin{vmatrix} (1,\ 1), & (1,\ 2), & (1,\ 3), & (1,\ x_p) \\ (2,\ 1), & (2,\ 2), & (2,\ 3), & (2,\ x_p) \\ (3,\ 1), & (3,\ 2), & (3,\ 3), & (3,\ x_p) \\ (4,\ 1), & (4,\ 2), & (4,\ 3), & (4,\ x_p) \end{vmatrix}$$

Secondly, the line in question is to be shewn perpendicular to any of the first three variation axes, for instance to T_2, whose direction cosines are

$$(2, x_1) \,/\, [2],\ (2, x_2) \,/\, [2], \ldots,\ (2, x_n) \,/\, [2].$$

It has, therefore, to be shewn $\Sigma[1, 2, 3, 4 \mid 1, 2, 3, x_p](2, x_p)=0$, $p=1, 2, \ldots, n$.

But the above expression is equal to, omitting suffix to x,

$$\Sigma \begin{vmatrix} (1,1), & (1,2), & (1,3), & (1,x) \\ (2,1), & (2,2), & (2,3), & (2,x) \\ (3,1), & (3,2), & (3,3), & (3,x) \\ (4,1), & (4,2), & (4,3), & (4,x) \end{vmatrix} (2, x) = \begin{vmatrix} (1,1), & (1,2), & (1,3), & \Sigma(1,x)\,(2,x) \\ (2,1), & (2,2), & (2,3), & \Sigma(2,x)\,(2,x) \\ (3,1), & (3,2), & (3,3), & \Sigma(3,x)\,(2,x) \\ (4,1), & (4,2), & (4,3), & \Sigma(4,x)\,(2,x) \end{vmatrix}$$

$$= \begin{vmatrix} (1,1), & (1,2), & (1,3), & (1,2) \\ (2,1), & (2,2), & (2,3), & (2,2) \\ (3,1), & (3,2), & (3,3), & (3,2) \\ (4,1), & (4,2), & (4,3), & (4,2) \end{vmatrix} = 0.$$

It may be noticed here that by the above method it follows

$$\Sigma[1,2,3,4 \mid 1,2,3,x](4,x) = \begin{vmatrix} (1,1), & (1,2), & (1,3) & (1,4) \\ (2,1), & (2,2), & (2,3) & (2,4) \\ (3,1), & (3,2), & (3,3) & (3,4) \\ (4,1), & (4,2), & (4,3) & (4,4) \end{vmatrix} = [1,2,3,4]^2.$$

Lastly, to shew $\Sigma[1,2,3,4 \mid 1,2,3,x]^2=[1,2,3]^2[1,2,3,4]^2$

This is easily seen to follow, for

$$\Sigma[1,2,3,4 \mid 1,2,3,x]^2 = \Sigma \begin{vmatrix} (1,1), & (1,2), & (1,3), & (1,x) \\ (2,1), & (2,2), & (2,3), & (2,x) \\ (3,1), & (3,2), & (3,3), & (3,x) \\ (4,1), & (4,2), & (4,3), & (4,x) \end{vmatrix} [1,2,3,4 \mid 1,2,3,x]$$

$$= \begin{vmatrix} (1,1), & (1,2), & (1,3), & \Sigma[1,2,3,4 \mid 1,2,3,x](1,x) \\ (2,1), & (2,2), & (2,3), & \Sigma[1,2,3,4 \mid 1,2,3,x](2,x) \\ (3,1), & (3,2), & (3,3), & \Sigma[1,2,3,4 \mid 1,2,3,x](3,x) \\ (4,1), & (4,2), & (4,3), & \Sigma[1,2,3,4 \mid 1,2,3,x](4,x) \end{vmatrix}$$

$$= \begin{vmatrix} (1,1), & (1,2), & (1,3), & [1,2,3,4 \mid 1,2,3,1] \\ (2,1), & (2,2), & (2,3), & [1,2,3,4 \mid 1,2,3,2] \\ (3,1), & (3,2), & (3,3), & [1,2,3,4 \mid 1,2,3,3] \\ (4,1), & (4,2), & (4,3), & [1,2,3,4 \mid 1,2,3,4] \end{vmatrix}$$

$$= [1,2,3]^2 [1,2,3,4]^2,$$

for $[1,2,3,4 \mid 1,2,3,1]=0$, $[1,2,3,4 \mid 1,2,3,2]=0$, $[1,2,3,4 \mid 1,2,3,3]=0$.

It may be interesting to note here, that by the above reasoning it can be shewn, that n lines whose direction cosines are

$(m_1, x_p)/[m_1], [m_1, m_2 \mid m_1, x_p]/[m_1][m_1, m_2], \ldots, [m_1, m_2, \ldots, m_n, \mid m_1, m_2, \ldots, m_{n-1}, x_p]/[m_1, m_2, \ldots, m_n]\,[m_1, m_2, \ldots, m_{n-1}]$, where $p=1, 2, \ldots, n$ and $m_1, m_2, \ldots, m_n$ are any distinct integers, will be mutually orthogonal.

In Paper I, it has been thought convenient to introduce a number of purely analytical definitions of the $n-1$ curvatures of a curve in n-dimensional space. These have been called the $n-1$ *principal curvatures* at a given point of the curve. They are not quite the same as the ordinarily defined curvatures, namely, the rates of turning of $E_1, E_2, \ldots, E_{n-1}$. These *principal curvatures* will be found more useful in practice. They are indicated by $u_1, u_2, \ldots, u_{n-1}$,

$$\text{where } u_1 = \frac{[1,2]}{[1]^3}, \quad u_2 = \frac{[1,2,3]}{[1,2]^2}, \quad u_3 = \frac{[1,2,3,4]}{[1,2,3]^2}\,\frac{[1,2]}{[1]}, \ldots,$$

$$u_{n-1} = \frac{[1,2,\ldots,n]\,[1,2,\ldots,n-2]}{[1,2,\ldots,n-1]^2[1]}$$

4. A General Theorem.

$$\text{Let } H_r = \frac{[1,2,\ldots,r \mid 1,2,\ldots,r-1,P]}{[1,2,\ldots,r][1,2,\ldots,r-1]}$$

$$\text{so that } H_{r+1} = \frac{[1,2,\ldots,r+1 \mid 1,2,\ldots,r,P]}{[1,2,\ldots,r+1][1,2,\ldots,r]}$$

$$\text{and } H_{r-1} = \frac{[1,2,\ldots,r-1 \mid 1,2,\ldots,r-2,P]}{[1,2,\ldots,r-1][1,2,\ldots,r-2]}$$

then, *whatever P may be*,

$$H_{r+1}u_r = H'_r + H_{r-1}u_{r-1},$$

where $H'_r = \frac{d}{ds}H_r$, and u_{r-1} and u_r are the $r-1$th and rth *principal curvatures*.

To prove this general theorem, it may be observed that

$$H_{r+1}[1,2,\ldots,r+1][1,2,\ldots,r]=[1,2,\ldots,r+1 \mid 1,2,\ldots,r,P]$$

$$=\begin{vmatrix} (1,1), & (1,2),\ldots\ldots,(1,r), & (1,P) \\ (2,1), & (2,2),\ldots\ldots,(2,r), & (2,P) \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ (r,1), & (r,2),\ldots\ldots,(r,r), & (r,P) \\ (r+1,1), & (r+1,2),\ldots,(r+1,r), & (r+1,P) \end{vmatrix} = \Delta_{r+1}, \text{ say}$$

$$=\frac{1}{[1,2,\ldots,r-1]^2}\begin{vmatrix} (1,1), & (1,2),\ldots\ldots,(1,r-1), & (1,r), & (1,P) \\ (2,1), & (2,2),\ldots\ldots,(2,r-1), & (2,r), & (2,P) \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots \\ (r,1), & (r,2),\ldots,(r,r-1), & (r,r), & (r,P) \\ 0\ , & 0\ ,\ldots,-[1,2,\ldots,r]^2, & [1,2,\ldots,r-1,r+1 \mid 1,2,\ldots,r], & D\ [1,2,\ldots,r \mid 1,2,\ldots,r-1,P] \end{vmatrix}$$

For, $D[1,2,\ldots, r \mid 1,2,\ldots, r-1, P] = [1,2,\ldots, r-1, r+1 \mid 1,2,\ldots, r-1,P]$

$$+[1,2,\ldots,r-1,r \mid 1,2,\ldots,r-2,r,P]$$

$$=\begin{vmatrix} (1,1), & (1,2),\ldots\ldots,(1,r-1), & (1,P) \\ (2,1), & (2,2),\ldots\ldots,(2,r-1), & (2,P) \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ (r-1,1), & (r-1,2),\ldots,(r-1,r-1), & (r-1,P) \\ (r+1,1), & (r+1,2),\ldots,(r+1,r-1), & (r+1,P) \end{vmatrix}$$

$$+\begin{vmatrix} (1,1), & (1,2),\ldots\ldots,(1,r-2), & (1,r), & (1,P) \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots \\ (r-2,1), & (r-2,2),\ldots,(r-2,r-2), & (r-2,r), & (r-2,P) \\ (r-1,1), & (r-1,2),\ldots,(r-1,r-2), & (r-1,r), & (r-1,P) \\ (r,1), & (r,2),\ldots\ldots,(r,r-2), & (r,r), & (r,P). \end{vmatrix}$$

$=[1,2,\ldots,r-1]^2(r+1,P)+A_r(r,P)+A_{r-1}(r-1,P)\ldots\ldots+A_1(1,P)$, where $A_r, A_{r-1},\ldots, A_1$ are independent of P.

Also replacing P by $r, r-1, r-2, \ldots, 2, 1$, the expression $[1, 2, \ldots, r-1, r+1 \mid 1, 2, \ldots, r-1, P] + [1, 2, \ldots, r-1, r \mid 1, 2 \ldots r-2, r, P]$ becomes $[1, 2 \ldots r-1, r+1 \mid 1, 2 \ldots r-1, r], -[1, 2, \ldots r-1, r]^2, 0, \ldots, 0, 0$, respectively.

Thus the transformation in the last row of $\triangle_{r+1}$ is proved.

Again, because

$[1, 2, \ldots, r \mid 1, 2, \ldots, r-1, P] = [1, 2 \ldots, r-1]^2 (r, P) + B_{r-1}(r-1, P) + \ldots + B_1 (1, P)$ where $B_{r-1}, \ldots, B_1$ are independent of P, and the expression $[1, 2, \ldots, r \mid 1, 2, \ldots, r-1, P]$, when P is replaced by $r, r-1, r-2, \ldots, 2, 1$, becomes

$$[1, 2, \ldots, r]^2, 0, 0, \ldots, 0, 0, \text{ respectively,}$$

the expression for $\triangle_{r+1}$ becomes

$$\frac{1}{[1, 2, \ldots, r-1]^4} \left| \begin{array}{llll} (1, 1), & (1, 2), & \ldots & (1, r-1), \\ \multicolumn{4}{c}{\dotfill} \\ (r-1, 1), & (r-1, 2), & \ldots, & (r-1, r-1), \\ 0, & 0, & & 0 \\ 0, & 0, & & -[1, 2, \ldots, r]^2, \end{array} \right.$$

$$\left. \begin{array}{cc} (1, r), & (1, P) \\ (r-1, r), & (r-1, P) \\ [1, 2, \ldots r]^2 & [1, 2, \ldots r \mid 1, 2, \ldots, r-1, P] \\ [1, 2, \ldots, r-1, r+1 \mid 1, 2, \ldots, r], & D[1, 2, \ldots, r \mid 1, 2, \ldots, r-1, P] \end{array} \right|$$

By repetitions of similar transformations, $\triangle_{r+1}$ reduces to

$$\left| \begin{array}{llllll} [1]^2, & (1, 2), & (1, 3) & \multicolumn{3}{l}{\dotfill} \\ 0, & [1, 2]^2, & [1, 2 \mid 1, 3] & \multicolumn{3}{l}{\dotfill} \\ 0, & 0, & [1, 2, 3]^2 & \multicolumn{3}{l}{\dotfill} \\ \multicolumn{6}{c}{\dotfill} \\ 0, & 0, & 0 & , \ldots, 0, & & [1, 2, \ldots, r-1]^2, \\ 0, & 0, & 0 & , \ldots, 0, & & 0, \\ 0, & 0, & 0 & , \ldots, 0, & & -[1, 2, \ldots, r]^2, \end{array} \right.$$

$$
\left.\begin{array}{r}
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1, P) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots[1, 2 \mid 1, P] \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots[1, 2, 3 \mid 1, 2, P] \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
[1, 2,\ldots, r-1 \mid 1, 2,\ldots, r-2, r],\ [1, 2,\ldots, r-1 \mid 1, 2,\ldots, r-2, P] \\
[1, 2,\ldots, r]^2,\quad [1, 2,\ldots, r \mid 1, 2,\ldots, r-1, P] \\
[1, 2,\ldots, r-1, r+1] \mid 1, 2,\ldots, r],\ D[1, 2,\ldots, r \mid 1, 2,\ldots, r-1, P.]
\end{array}\right|
$$

divided by $[1]^2[1, 2]^2 \ldots [1, 2,\ldots, r-2]^2[1, 2,\ldots, r-1]^4$.

$$
\text{Or, } \triangle_{r+1} = [1,2,\ldots, r-1]^{-4} \left|\begin{array}{lll}
[1,2,\ldots,r-1]^2, & [1,2,\ldots,r-1 \mid 1,2 \ldots,r-2,r], & [1, 2,\ldots, r-1 \mid 1, 2,\ldots, r-2, P] \\
0, & [1, 2,\ldots,r]^2 & [1,2,\ldots,r \mid 1, 2,\ldots,r-1, P] \\
-[1,2,\ldots,r]^2, & [1,2,\ldots,r-1,r+1 \mid 1,2,\ldots,r], & D[1,2,\ldots,r \mid 1, 2,\ldots,r-1, P]
\end{array}\right|
$$

$$
\text{Or, } H_{r+1} \frac{[1, 2,\ldots, r+1][1,2 \ldots,r-1]}{[1, 2,\ldots, r]^2} = \frac{D[1, 2,\ldots, r \mid 1,2,\ldots, r-1,P]}{[1,2,\ldots,r][1,2,\ldots,r-1]}
$$

$$
-\frac{[1, 2,\ldots, r \mid 1, 2,\ldots, r-1, P]}{[1, 2,\ldots, r]^2[1,2,\ldots,r-1]^2} D\{[1, 2,\ldots, r][1, 2,\ldots, r-1]\}
$$

$$
+\frac{[1, 2,\ldots, r][1\cdot 2,\ldots, r-2]}{[1, 2,\ldots, r-1]^2} \cdot \frac{[1, 2,\ldots, r-1 \mid 1, 2,\ldots, r-2, P]}{[1, 2,\ldots, r-1][1, 2,\ldots, r-2]}
$$

$$
= D\frac{[1, 2,\ldots, r \mid 1, 2,\ldots, r-1, P]}{[1, 2,\ldots,r \mid 1, 2,\ldots, r-1\ ]} + \frac{[1, 2,\ldots, r][1, 2,\ldots, r-2]}{[1, 2,\ldots, r-1]^2}
$$

$$
\times \frac{[1, 2,\ldots, r-1 \mid 1, 2,\ldots, r-2, P]}{[1, 2,\ldots, r-1][1, 2,\ldots, r-2]}
$$

Or, dividing by [1]

$$
H_{r+1} \cdot \frac{[1,2,\ldots,r+1][1,2,\ldots,r-1]}{[1, 2,\ldots,r]^2[1]} = \frac{DH_r}{[1]} + \frac{[1,2 \ldots,r][1, 2,\ldots, r-2]}{[1, 2,\ldots, r-1]^2[1]} H_{r-1},
$$

Or, $H_{r+1}\, u_r = \frac{d}{ds} H_r + H_{r-1}\, u_{r-1} = H'_r + H_{r-1}\, u_{r-1}$, which proves the theorem.

5. Extension of Serret-Frenet Formulae.

Let $\lambda_{p,\,1}, \lambda_{p,\,2}, \ldots, \lambda_{p,\,n}$ be the cosines of the angles which the pth principal axis makes with the n co-ordinate axes, so that $\lambda_{p,\,r}$ is the cosine of the angle which λ_p makes with the axis of x_r. Then

$$\lambda_{1,\,r} = \frac{(1,\ x_r)}{[1]},$$

$$\lambda_{2,\,r} = \frac{[1,\,2 \mid 1,\,x_r]}{[1][1,\,2]},$$

$$\lambda_{3,\,r} = \frac{[1,\,2,\,3 \mid 1,\,2,\,x_r]}{[1,\,2][1,\,2,\,3]}, \text{ etc.}$$

$$\lambda_{n,\,r} = \frac{[1,\,2,\ldots,\,n \mid 1,\,2,\ldots,\,n-1,\ x_r]}{[1,\,2,\ldots,\,n-1][1,\,2,\ldots,\,n]}$$

Hence, from the general theorem of last article, putting x_r for P, it follows that

$$\frac{d}{ds}\lambda_{q,\,r} = \lambda_{q+1,\,r}\,u_q - \lambda_{q-1,\,r}\,u_{q-1}.$$

Hence, observing that $u_n = 0$ and

$$\frac{d\lambda_{1,\,r}}{ds} = \frac{d}{ds}\frac{(1,\ x_r)}{[1]} = [1]^{-1}\,D\frac{(1,\ x_r)}{[1]}$$

$$= [1]^{-1}\left\{\frac{(2,\ x_r)}{[1]} - \frac{(1,\ x_r)(1,\,2)}{[1]^3}\right\}$$

$$= \frac{[1,\,2]}{[1]^3}\,\frac{[1,\,2 \mid 1,\ x_r]}{[1,\,2][1]} = u_1\lambda_{2,\,r}$$

the following series of formulæ may be written down:

$$\frac{d\lambda_{1,\,r}}{ds} = \lambda'_{1,\,r} = u_1\lambda_{2,\,r}$$

$$\lambda'_{2,\,r} = u_2\lambda_{3,\,r} - u_1\lambda_{1,\,r}$$

$$\lambda'_{3,\,r}=u_3\lambda_{4,\,r}-u_2\lambda_{2,\,r}$$

....

$$\lambda'_{n-1,\,r}=u_{n-1}\,\lambda_{n,\,r}-u_{n-2}\,\lambda_{n-2,\,r}$$

$$\lambda'_{n,\,r}=-u_{n-1,\,r}\,\lambda_{n-1,\,r}$$

Professor J. G. Hardy, in the American Journal of Mathematics, Vol. XXIV, page 4, deduces extensions of Frenet's Formulæ to curves in four-dimensional space. His results come out in slightly different and less simple forms, as he does not conceive principal curvatures.

PARAMETRIC CO-EFFICIENTS IN THE DIFFERENTIAL GEOMETRY OF CURVES IN AN N-SPACE

BY

SYAMADAS MUKHOPADHYAYA.

III. FUNDAMENTAL FORMULAE.*

1. Notational Extensions.

We have already denoted by (p, q) the expression

$$\Sigma x_r^{(p)} x_r^{(q)}, \; r=1, 2, \dots, n,$$

where $x_1, x_2, \dots, x_n$ are the n orthogonal co-ordinates of a point on a curve in an n-space, considered as functions of a single parameter t, and (p), (q) indicate differentiations with respect to t, p times and q times, respectively.

If we consider our formulæ from a purely algebraical point of view, independent of any notion of differentiation or of geometrical application, we may look upon $x_1, x_2, \dots, x_n$ as any n algebraical quantities, (p) and (q) being only distinguishing suffixes. In the first five articles of this paper we will adopt this extended definition of (p, q).

With the extension of meaning of (p, q) all the other symbolical expressions we have brought into use will have similar extensions of meaning.

Thus $[l, m \mid p, q]$, $[l, m, n \mid p, q, r]$, etc., will simply stand for

$$\begin{vmatrix} (l, p), & (l, q) \\ (m, p), & (m, q) \end{vmatrix}, \quad \begin{vmatrix} (l, p), & (l, q), & (l, r) \\ (m, p), & (m, q), & (m, r) \\ (n, p), & (n, q), & (n, r) \end{vmatrix}, \text{ etc.,}$$

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* From *Bulletin, Calcutta Mathematical Society*, Vol. 2, 1910.

and $[l]^2$, $[l, m]^2$, $[l, m, n]^2$, etc., will stand simply for

$$(l, l),\ \begin{vmatrix}(l, l), & (l, m)\\ (m, l), & (m, m)\end{vmatrix},\ \begin{vmatrix}(l, l), & (l, m), & (l, n)\\ (m, l), & (m, m), & (m, n)\\ (n, l), & (n, m), & (n, n)\end{vmatrix},\ \text{etc.}$$

Similarly, (l, P), $[l, m \mid p, P]$, $[l, m, n \mid p, q, P]$, $[l, m, n \mid p, P, Q]$, etc., will simply stand for

$$P^{(l)},\ \begin{vmatrix}(l, p), & P^{(l)}\\ (m, p), & P^{(m)}\end{vmatrix},\ \begin{vmatrix}(l, p), & (l, q), & P^{(l)}\\ (m, p), & (m, q), & P^{(m)}\\ (n, p), & (n, q), & P^{(n)}\end{vmatrix},\ \begin{vmatrix}(l, p), & P^{(l)}, & Q^{(l)}\\ (m, p), & P^{(m)}, & Q^{(m)}\\ (n, p), & P^{(n)}, & Q^{(n)}\end{vmatrix},\ \text{etc.},$$

where $P^{(l)}$, $P^{(m)}$, $P^{(n)}$, $Q^{(l)}$, $Q^{(m)}$, $Q^{(n)}$, etc., may stand for any defined sets of quantities.

Further, $[l, P \mid p, Q]$, $[l, m, P \mid p, q, Q]$, etc., will stand simply for

$$-\begin{vmatrix}(l, p), & Q^{(l)}\\ P^{(p)}, & O\end{vmatrix},\ -\begin{vmatrix}(l, p), & (l, q), & Q^{(l)}\\ (m, p), & (m, q), & Q^{(m)}\\ P^{(p)}, & P^{(q)}, & O\end{vmatrix},\ \text{etc.}$$

2. The Symbol Δ_r and Δ-transformations.

We will now introduce a new symbol Δ_r which will have a fundamental place in our investigations. It is defined as follows:

$$\Delta_1(l) = \frac{(1, l)}{[1]},\ \Delta_2(l) = \frac{[1, 2 \mid 1, l]}{[1, 2][1]},\ \Delta_3(l) = \frac{[1,2,3 \mid 1,2,l]}{[1,2,3][1,2]},\ \text{\&c.},$$

... ...

$$\Delta_r(l) = \frac{[1,2,\ldots,r \mid 1,2,\ldots,r-1,l]}{[1,2,\ldots,r]\,[1,2,\ldots,r-1]},\ \Delta_1(P) = \frac{(1,P)}{[1]},$$

$$\Delta_2(P) = \frac{[1,2 \mid 1,P]}{[1,2]\,[1]},\ \Delta_3(P) = \frac{[1,2,3 \mid 1,2,P]}{[1,2,3]\,[1,2]},\ \text{etc.},$$

$$\Delta_r(P) = \frac{[1,2,\ldots,r \mid 1,2,\ldots,r-1,P]}{[1,2,\ldots,r]\,[1,2,\ldots,r-1]},$$

Further the symbol Δ_r (Δ_s) will denote

$$\frac{[1,2,..,r \mid 1,2,..,r-1,\ \Delta_s]}{[1,2,\ldots,r]\ [1,2,\ldots\ r-1]}$$

which again stands for

$$\begin{vmatrix} (1,1), & (1,2),\ldots, & (1,r-1), & \Delta_s(1) \\ (2,1), & (2,2),\ldots, & (2,r-1), & \Delta_s(2) \\ \cdots & \cdots & \cdots & \cdots \\ (r-1,1), & (r-1,2),\ldots, & (r-1,r-1), & \Delta_s(r-1) \\ (r,1), & (r,2),\ldots, & (r,r-1), & \Delta_s(r) \end{vmatrix} \div [1,2,..,r][1,2,r-1]$$

If in the last determinant we change any column, say the l^{th}, which is not the last, into $\Delta_l(1)$, $\Delta_l(2),\ldots,$ $\Delta_l(r)$, the value of the determinant will be altered in the ratio

$$\frac{[1,2,\ldots,l-1]}{[1,2,\ldots,l]},$$

$$\text{for } \Delta_l(r) = \frac{[1,2,..,l \mid 1,2,..,l-1,\ r]}{[1,2,..l]\ [1,2,..,l-1]}$$

$$= \begin{vmatrix} (1,1), & (1,2),\ldots, & (1,l-1), & (1,r) \\ (2,1), & (2,2),\ldots, & (2,l-1), & (2,r) \\ \cdots & \cdots & \cdots & \cdots \\ (l,1), & (l,2),\ldots, & (l,l-1), & (l,r) \end{vmatrix} \div [1,2,..,l-1]\ [1,2,..,l]$$

$$= (l,r)\ \frac{[1,2,..,l-1]}{[1,2,..,l]} + (l-1,r)\ A_{l-1} + (l-2,r)\ A_{l-2} + \ldots + (1,r)A_1,$$

where $A_1, A_2, \ldots A_{l-1}$ are co-efficients which do not involve r.

In fact the operation, which we will call a Δ-operation, is equivalent to multiplying the l^{th} column by $\dfrac{[1,2,..,l-1]}{[1,2,..,l]}$ and adding to it the $l-1$ preceding columns multiplied by A_1, A_2, ..., A_{l-1}, respectively.

Hence $\Delta_r(\Delta_s)$

$$=\begin{vmatrix} (1,1), (1,2), \ldots (1, r-2), \Delta_{r-1}(1), \Delta_s(1) \\ (2,1), (2,2), \ldots (2, r-2), \Delta_{r-1}(2), \Delta_s(2) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (r,1), (r,2), \ldots (r, r-2), \Delta_{r-1}(r) \Delta_s(r) \end{vmatrix} \div [1, 2, .., r-2][1, 2, .., r]$$

$$=\begin{vmatrix} \Delta_1(1), \Delta_2(1), \ldots \Delta_{r-1}(1), \Delta_s(1) \\ \Delta_1(2), \Delta_2(2), \ldots \Delta_{r-1}(2), \Delta_s(2) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta_1(r), \Delta_2(r), \ldots \Delta_{r-1}(r), \Delta_s(r) \end{vmatrix} \div [1, 2, .., r],$$

after successive Δ-transformations of the first $r-1$ columns. In fact the result of successive Δ-transformations of the first l columns or rows of a suitable determinant is equivalent to dividing the determinant by $[1, 2, .., l]$.

It is easy to see that $\Delta_r(\Delta_s)$ vanishes when r is not equal to s and is equal to unity when r is equal to s. For if s be less than r, then two columns in the last determinant become identical and if s be greater than r then $\Delta_s(1), \Delta_s(2), \ldots \Delta_s(r)$ all vanish from definition.

If s be equal to r, then

$$\Delta_r(\Delta_s) = \Delta_1(1)\,\Delta_2(2)\,..\,\Delta_r(r) \div [1, 2, .., r]$$

$$= [1] \,.\, \frac{[1, 2]}{[1]} \,.\,.\,.\, \frac{[1, 2, .., r]}{[1, 2, .., r-1]} \div [1, 2, .., r] = 1.$$

We have $\Sigma \mathrm{x}_r^{(q)} \Delta_p(\mathrm{x}_r)$, $r = 1, 2, \ldots, n$,

$$=\begin{vmatrix} (1,1), (1,2), .. (1, p-1), \Sigma \mathrm{x}_r^{(1)} \mathrm{x}_r^{(q)} \\ (2,1), (2,2), .. (2, p-1), \Sigma \mathrm{x}_r^{(2)} \mathrm{x}_r^{(q)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (p,1), (p,2), .. (p, p-1), \Sigma \mathrm{x}_r^{(p)} \mathrm{x}_r^{(q)} \end{vmatrix} \div [1, 2, .., p][1, 2, .., p-1]$$

$$=\begin{vmatrix} (1,1), (1,2), .. (1, p-1), (1, q) \\ (2,1), (2,1), .. (2, p-1), (2, q) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (p,1), (p,2), .. (p, p-1), (p, q) \end{vmatrix} \div [1, 2, .., p][1, 2, .., p-1]$$

$$=[1, 2, .., p \mid 1, 2, .., p-1, q] \div [1, 2, .., p][1, 2, .., p-1]$$

That is, $\Sigma x_r^{(q)} \Delta_p(x_r) = \Delta_p(q)$ (α)

Again $\Sigma \Delta_p(x_r)\, \Delta_q(x_r)$, $r=1, 2, .., n$,

$$=\begin{vmatrix} (1, 1), (1, 2), .., (1, p-1), \Sigma x^{(1)} \Delta_q(x_r) \\ (2, 1), (2, 2), .., (2, p-1), \Sigma x^{(2)} \Delta_q(x_r) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (p, 1), (p, 2), .., (p, p-1), \Sigma x^{(p)} \Delta_q(x_r) \end{vmatrix} \div [1, 2, .., p][1, 2, .., p-1]$$

$$=\begin{vmatrix} (1, 1), (1, 2), .., (1, p-1), \Delta_q(1) \\ (2, 1), (2, 1), .., (2, p-1), \Delta_q(2) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (p, 1), (p, 2), .., (p, p-1), \Delta_q(p) \end{vmatrix} \div [1, 2, .., p][1, 2, .., p-1]$$

$$=[1, 2, .., p \mid 1, 2, .., p-1, \Delta_q] \div [1, 2, .., p][1, 2, .., p-1]$$

That is $\Sigma \Delta_p(x_r)\, \Delta_q(x_r) = \Delta_p(\Delta_q) = 1$ or 0, (β)

according as p is equal or not equal to q.

3. A fundamental Identity.

We will now establish the following fundamental identity:

$$\frac{[1, 2, 3, ..., r, \mathrm{P} \mid 1, 2, 3, ..., r, \mathrm{Q}]}{[1, 2, 3, ..., r]^2}$$

$$=\Delta_1(\mathrm{P})\Delta_1(\mathrm{Q}) + \Delta_2(\mathrm{P})\Delta_2(\mathrm{Q}) + \;..+ \Delta_r(\mathrm{P})\Delta_r(\mathrm{Q}) \qquad \text{(A)}$$

We have, by definition, $[1, 2, .., r, \mathrm{P} \mid 1, 2, .., r, \mathrm{Q}]$

$$=-\begin{vmatrix} (1, 1), (1, 2), ..., (1, r), \mathrm{Q}^{(1)} \\ (2, 1), (2, 2), ..., (2, r), \mathrm{Q}^{(2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (r, 1), (r, 2), ..., (r, r), \mathrm{Q}^{(r)} \\ \mathrm{P}^{(1)}, \mathrm{P}^{(2)}, ..., \mathrm{P}^{(r)}, 0 \end{vmatrix} \qquad (i)$$

Therefore, $\dfrac{[1, 2, .., r, \mathrm{P} \mid 1, 2, .., r, \mathrm{Q}]}{[1, 2, .., r]}$

$$= -\begin{vmatrix} \triangle_1(1), & \triangle_2(1), & .. & \triangle_r(1), & \mathrm{Q}^{(1)} \\ \triangle_1(2), & \triangle_2(2), & .. & \triangle_r(2), & \mathrm{Q}^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \triangle_1(r), & \triangle_2(r), & .. & \triangle_r(r), & \mathrm{Q}^{(r)} \\ \triangle_1(\mathrm{P}), & \triangle_2(\mathrm{P}), & .. & \triangle_r(\mathrm{P}), & 0 \end{vmatrix} \qquad (ii)$$

by $\triangle$-transformations of the first r columns of the determinant (*i*).

Again $\dfrac{[1, 2, \,.\,.\,, r, P, \mid 1, 2, \,.\,.\,, r, Q]}{[1, 2, .., r]^2}$

$$= -\begin{vmatrix} \triangle_1(\triangle_1), & \triangle_1(\triangle_2), & .\,.\,, & \triangle_1(\triangle_r), & \triangle_1(\mathrm{Q}) \\ \triangle_2(\triangle_1), & \triangle_2(\triangle_2), & .\,.\,, & \triangle_2(\triangle_r), & \triangle_2(\mathrm{Q}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \triangle_r(\triangle_1), & \triangle_r(\triangle_2), & .\,.\,, & \triangle_r(\triangle_r), & \triangle_r(\mathrm{Q}) \\ \triangle_1(\mathrm{P}), & \triangle_2(\mathrm{P}), & .., & \triangle_r(\mathrm{P}), & 0 \end{vmatrix}$$

by $\triangle$-transformations of the first r rows of determinant (*ii*),

$$= -\begin{vmatrix} 1 & , 0 & , \,.\,.\,, & 0 & , \triangle_1(\mathrm{Q}) \\ 0 & , 1 & , \,.\,.\,, & 0 & , \triangle_2(\mathrm{Q}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & , 0 & , \,.\,.\,, & 1 & , \triangle_r(\mathrm{Q}) \\ \triangle_1(\mathrm{P}), & \triangle_2(\mathrm{P}), & .\,.\,, & \triangle_r(\mathrm{P}), & 0 \end{vmatrix}$$

$$= \triangle_1(\mathrm{P})\, \triangle_1(\mathrm{Q}) + \triangle_2(\mathrm{P})\, \triangle_2(\mathrm{Q}) + \ldots + \triangle_r(\mathrm{P})\, \triangle_r(\mathrm{Q}).$$

If in identity (A) we put Q equal to P, we have

$$\frac{[1, 2, \,.\,.\,, r, \mathrm{P}]^2}{[1, 2, \,.\,.\,, r]^2} = \triangle_1^2(\mathrm{P}) + \triangle_2^2(\mathrm{P}) + .. + \triangle_r^2(\mathrm{P}) \qquad (\mathrm{A}_1)$$

where $\triangle_r^2(\mathrm{P})$ stands for $[\Delta_r(\mathrm{P})]^2$ and $[1, 2, \,.\,.\,, r, \mathrm{P}]^2$ for $[1, 2, .., r, \mathrm{P} \mid 1, 2, \,.\,.\,, r, \mathrm{P}]$.

4. Derived Identities.

We have, by definition, $[1, 2, .., r, P \mid 1, 2, .., r, Q]$

$$= -\begin{vmatrix} (1, 1), & (1, 2), & .\ . & (1, r), & Q^{(1)} \\ (2, 1), & (2, 2), & .\ . & (2, r), & Q^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (r, 1), & (r, 2), & .\ . & (r, r), & Q^{(r)} \\ P^{(1)}, & P^{(2)}, & .\ . & P^{(r)}, & 0 \end{vmatrix} \qquad (iii)$$

If we write $(1, s)$, $(2, s)$, . . (r, s) for $Q^{(1)}$, $Q^{(2)}$, . . $Q^{(r)}$, then determinant (*iii*) becomes

$$-\begin{vmatrix} (1, 1), & (1, 2), & .\ . & (1, r), & (1, s) \\ (2, 1), & (2, 2), & .\ . & (2, r), & (2, s) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (r, 1), & (r, 2), & .\ . & (r, r), & (r, s) \\ P^{(1)}, & P^{(2)}, & .\ . & P^{(r)}, & 0 \end{vmatrix}$$

$$= P^{(s)}\begin{vmatrix} 1, 1), & (1, 2), & .\ . & (1, r) \\ (2, 1), & (2, 2), & .\ . & (2, r) \\ \cdots & \cdots & \cdots & \cdots \\ (r, 1), & (r, 2), & .\ . & (r, r) \end{vmatrix} - \begin{vmatrix} (1, 1), & (1, 2), & .\ . & (1, r), & (1, s) \\ (2, 1), & (2, 2), & .\ . & (2, r), & (2, s) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (r, 1), & (r, 2), & .\ . & (r, r), & (r, s) \\ P^{(1)}, & P^{(2)}, & .\ . & P^{(r)}, & P^{(s)} \end{vmatrix}$$

$$= P^{(s)}[1, 2, .\ .\ .\ , r]^2 - [1, 2, .\ .\ , r, s \mid 1, 2, .\ .\ .\ , r, P]$$

Also, by definition, $[1, 2, .\ .\ r \mid 1, 2, .\ .\ r-1, Q]$

$$= \begin{vmatrix} (1, 1), & (1, 2), & .\ . & (1, r-1), & Q^{(1)} \\ (2, 1), & (2, 2), & .\ . & (2, r-1), & Q^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (r, 1), & (r, 2), & .\ . & (r, r-1), & Q^{(r)} \end{vmatrix} \qquad (iv)$$

If we write $(1, s), (2, s), \ldots (r, s)$, for $Q^{(1)}, Q^{(2)}, \ldots Q^{(r)}$, then determinant (*iv*) becomes

$$\begin{vmatrix} (1, 1), (1\ 2), \ldots, (1, r-1), (1, s) \\ (2, 1), (2, 1), \ldots, (2, r-1), (2, s) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (r, 1), (r, 2), \ldots, (r, r-1), (r, s) \end{vmatrix}$$

$$=[1, 2, \ldots, r \mid 1, 2, \ldots, r-1, s],$$

that is, Δ_r (Q) becomes Δ_r (s), and identity (A) becomes

$$P^{(s)} - \frac{[1, 2, \ldots, r, s \mid 1, 2, \ldots, r, P]}{[1, 2, \ldots, r]^2}$$

$$=\Delta_1(s)\Delta_1(P)+\Delta_2(s)\Delta_2(P)+\ldots+\Delta_r(s)\Delta_r(P) \qquad \text{(B)}$$

If in identity (B) we put $r=s$, then we have

$$P^{(s)}=\Delta_1(s)\Delta_1(P)+\Delta_2(s)\Delta_2(P)+\ldots+\Delta_s(s)\Delta_s(P) \qquad (B_1)$$

If we subtract (B) from (B_1), we have $\dfrac{[1, 2, \ldots, r, s \mid 1, 2, \ldots, r, P)}{[1, 2, \ldots, r]^2}$

$$=\Delta_{r+1}(s)\Delta_{r+1}(P)+\Delta_{r+2}(s)\Delta_{r+2}(P)+\ldots+\Delta_s(s)\Delta_s(P) \qquad (B_2)$$

Again if in (*iii*) we put $(1, p), (2, p) \ldots (r, p), (1, q), (2, q), \ldots (r, q)$ for $P^{(1)}, P^{(2)}, \ldots P^{(r)}, Q^{(1)}, Q^{(2)}, \ldots Q^{(r)}$, respectively, then (*iii*) becomes

$$\begin{vmatrix} (1, 1), (1, 2), \ldots, (1, r), (1, q) \\ (2, 1), (2, 2), \ldots (2, r), (2, q) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (r, 1), (r, 2), \ldots (r, r), (r, q) \\ (p, 1), (p, 2), \ldots (p, r), 0 \end{vmatrix}$$

$$=(p, q) \begin{vmatrix} (1, 1), (1, 2), \ldots (1, r) \\ (2, 1), (2, 2), \ldots (2, r) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (r, 1), (r, 2), \ldots (r, r) \\ \end{vmatrix} - \begin{vmatrix} (1, 1), (1, 2), \ldots (3, r), (1, q) \\ (2, 1), (2, 2), \ldots (2, r), (2, q) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (r, 1), (r, 2), \ldots (r, r), (r, q) \\ (p, 1), (p, 2), \ldots (p, r), (p, q) \end{vmatrix}$$

$$=(p, q)\ [1, 2, \ldots, r]^2-[1, 2, \ldots, r, p \mid 1, 2, \ldots, r, q]$$

Also $\Delta_r(P)$ and $\Delta_r(Q)$ become $\Delta_r(p)$ and $\Delta_r(q)$, respectively.

Hence from identity (A), we have

$$(p, q) - \frac{[1, 2, .., r, p \mid 1, 2, .., r, q]}{[1, 2, .., r]^2} = \Delta_1(p)\,\Delta_1(q) + \Delta_2(p)\Delta_2(q)$$

$$+ .. + \Delta_r(p)\,\Delta_r(q) \qquad (C)$$

If in (C) we suppose p not greater than q and put $r=p$, we have

$$(p, q) = \Delta_1(p)\Delta_1(q) + \Delta_2(p)\Delta_2(q) + .. + \Delta_p(p)\Delta_p(q) \qquad (C_1)$$

If in (C_1) we put $q=p$, we have,

$$(p, p) = \Delta_1^2(p) + \Delta_2^2(p) + .. + \Delta_p^2(p) \qquad (C_2)$$

If we subtract (C) from (C_1), we have

$$\frac{[1, 2, .., r, p \mid 1, 2, .., r, q]}{[1, 2, .., r]^2} = \Delta_{r+1}(p)\Delta_{r+1}(q) + \Delta_{r+2}(p)\Delta_{r+2}(q) +$$

$$\Delta_p(p)\Delta_p(q) \qquad (C_3)$$

where r is supposed less than p and p not greater than q.

If in (C_3) we put $q=p$, then we have

$$\frac{[1, 2, .., r, p]^2}{[1, 2, .., r]^2} = \Delta^2_{r+1}(p) + \Delta^2_{r+2}(p) + .. + \Delta^2_p(p) \qquad (C_4)$$

5. Three more Identities.

We will establish further the three following identities:

$$\frac{[1, 2, .., r \mid 1, 2, .., r-2, P, Q]}{[1, 2, .., r][1, 2, .., r-2]} = \Delta_{r-1}(P)\Delta_r(Q) - \Delta_{r-1}(Q)\Delta_r(P) \qquad (D)$$

$$\frac{[1, 2, .., r \mid 1, 2, .., r-2, p, Q]}{[1, 2, .., r][1, 2, .., r-2]} = \Delta_{r-1}(p)\Delta_r(Q) - \Delta_{r-1}(Q)\Delta_r(p) \qquad (D_1)$$

$$\frac{[1, 2, .., r \mid 1, 2, .., r-2, p, q]}{[1, 2, .., r][1, 2, .., r-2]} = \Delta_{r-1}(p)\Delta_r(q) - \Delta_{r-1}(q)\Delta_r(p) \qquad (D_2)$$

To prove (D), we have, by definition,

$$[1, 2, .., r \mid 1, 2, .., r-2, P, Q] = \begin{vmatrix} (1,1), (1,2), .., (1, r-2), P^{(1)}\ Q^{(1)} \\ (2,1), (2,2), .., (2,r-2), P^{(2)}, Q^{(2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (r,1), (r,2), .., (r, r-2), P^{(r)}, Q^{(r)} \end{vmatrix}$$

Therefore $$\frac{[1,2, .., r, \mid 1,2, .., r-2, P,Q]}{[1,2, .., r]\,[1,2, .., r-2]}$$

$$= \begin{vmatrix} \Delta_1(\Delta_1), & \Delta_1(\Delta_2), .., & \Delta_1(\Delta_{r-2}), & \Delta_1(P), & \Delta_1(Q) \\ \Delta_2(\Delta_1), & \Delta_2(\Delta_2), .., & \Delta_2(\Delta_{r-2}), & \Delta_2(P), & \Delta_2(Q) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta_{r-2}(\Delta_1), & \Delta_{r-2}(\Delta_2), .., & \Delta_{r-2}(\Delta_{r-2}), & \Delta_{r-2}(P), & \Delta_{r-2}(Q) \\ \Delta_{r-1}(\Delta_1), & \Delta_{r-1}(\Delta_2), .., & \Delta_{r-1}(\Delta_{r-2}), & \Delta_{r-1}(P), & \Delta_{r-1}(Q) \\ \Delta_r(\Delta_1), & \Delta_r(\Delta_2), .., & \Delta_r(\Delta_{r-2}), & \Delta_r(P), & \Delta_r(Q) \end{vmatrix}$$

by successive Δ-transformations of the r rows and the first $r-2$ columns,

$$= \begin{vmatrix} 1, 0, ..., 0, \Delta_1(P), & \Delta_1(Q) \\ 0, 1, ..., 0, \Delta_2(P), & \Delta_2(Q) \\ \cdots\cdots\cdots\cdots & \cdots \\ 0, 0, ..., 1, \Delta_{r-2}(P), & \Delta_{r-2}(Q) \\ 0, 0, ..., 0, \Delta_{r-1}(P), & \Delta_{r-1}(Q) \\ 0, 0, ..., 0, \nabla_r(P), & \Delta_r(Q) \end{vmatrix}$$

$$= \begin{vmatrix} \Delta_{r-1}(P), & \Delta_{r-1}(Q) \\ \Delta_r(P), & \Delta_r(Q) \end{vmatrix}$$

Identities (D_1) and (D_2) are established in the same way.

6. Fundamental Differentiation Formulae.

We will now establish the two following Fundamental Differentiation Formulæ:

$$\Delta'_r(P) = \Delta_{r+1}(P)\, u_r - \Delta_{r-1}(P)\, u_{r-1} \tag{E}$$

$$\Delta'_r(p) = \Delta_{r+1}(p)\, u_r + \Delta_r(p+1) - \Delta_{r-1}(p) u_{r-1} \tag{E$_1$}$$

Where we suppose $x_1, x_2, \ldots, x_n$ to be the n co-ordinates of a point on a curve in an n-space, $u_1, u_2, \ldots u_{n-1}$ the $n-1$ principal* curvatures of the curve at the given point, $x_1^{(r)}, x_2^{(r)}, \ldots, x_n^{(r)}$, $P^{(r)}$ the r^{th} differential co-efficients of $x_1, x_2, \ldots, x_n$, P respectively, considered as functions of the arcual distance s of the given point from an assumed point, and $\Delta'_r(P)$, $\Delta'_r(p)$ are the first derived of $\Delta_r(P)$, $\Delta_r(p)$, respectively with respect to s.

We have, by the laws of differentiation of parametric co-efficients,†

$$[1, 2, .., r \mid 1, 2, .., r-1, P]'$$

$$= [1, 2, .., r-1, r+1 \mid 1, 2, .., r-1, P]$$

$$+ [1, 2, .., r \mid 1, 2, .., r-2, r, P]$$

$$= [1, 2, .., r-1]^2 \{\Delta_r(r+1)\,\Delta_r(P) + \Delta_{r+1}(r+1)\,\Delta_{r+1}(P)\}$$

$$+ [1, 2, .., r][1, 2, .., r-2]\{\Delta_{r-1}(r)\,\Delta_r(P) - \Delta_r(r)\,\Delta_{r-1}(P)\}$$

by formulæ (B_2) and (D_1).

Therefore $\dfrac{[1, 2, .., r \mid 1, 2, .., r-1, P]'}{[1, 2, .., r][1, 2, .., r-1]}$

$$= \frac{\Delta_r(r+1)\,\Delta_r(P) + \Delta_{r+1}(r+1)\,\Delta_{r+1}(P)}{\Delta_r(r)}$$

$$+ \frac{\Delta_{r-1}(r)\,\Delta_r(P) - \Delta_r(r)\,\Delta_{r-1}(P)}{\Delta_{r-1}(r-1)}$$

$$= \frac{\Delta_{r+1}(r+1)}{\Delta_r(r)}\Delta_{r+1}(P) - \frac{\Delta_r(r)}{\Delta_{r-1}(r-1)}\Delta_{r-1}(P)$$

$$+ \left\{\frac{\Delta_r(r+1)}{\Delta_r(r)} + \frac{\Delta_{r-1}(r)}{\Delta_{r-1}(r+1)}\right\}\Delta_r(P)$$

$$= u_r\,\Delta_{r+1}(P) - u_{r-1}\,\Delta_{r-1}(P) + \left\{\frac{[1, 2, .., r \mid 1, 2, .., r-1, r+1]}{[1, 2, \cdot, r]^2}\right.$$

$$\left. + \frac{[1, 2, .., r-1 \mid 1, 2, .., r-2, r]}{[1, 2, .., r-1]^2}\right\}\Delta_r(P)$$

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[*] Vide *Bulletin, C.M.S.*, Vol. I, pp. 240-41.

[†] Vide *Bulletin, C.M.S.*, Vol. I, p. 247.

$$=\left\{\frac{[1,2,..,r]'}{[1,2,..,r]}+\frac{[1,2,..,r-1]'}{[1,2,..,r-1]}\right\}\Delta_r\,(\mathrm{P})+u_r\,\Delta_{r+1}\,(\mathrm{P})$$

$$-u_{r-1}\,\Delta_{r-1}\,(\mathrm{P})$$

$$=\frac{\{[1,2,\ldots,r][1,2,\ldots,r-1]\,\Delta_r\,(\mathrm{P})\}'}{[1,2,\ldots,r][1,2,\ldots,r-1]}-\Delta'_r\,(\mathrm{P})+u_r\,\Delta_{r+1}\,(\mathrm{P})$$

$$-u_{r-1}\Delta_{r-1}\,(\mathrm{P})$$

$$=\frac{[1,2,..,r\mid 1,2,..,r-1,\mathrm{P}]'}{[1,2,..,r][1,2,..,r-1]}-\Delta'_r\,(P)+u_r\Delta_{r+1}\,(P)$$

$$-u_{r-1}\Delta_{r-1}\,(\mathrm{P})$$

Therefore $\Delta'_r\,(\mathrm{P})=u_r\Delta_{r+1}\,(\mathrm{P})-u_{r-1}\Delta_{r-1}\,(\mathrm{P})$ (E)*

To prove formula (E_1) we have

$$[1,2,..,r\mid 1,2,..,r-1,p]'$$

$$=[1,2,..,r-1,r+1\mid 1,2,\ldots,r-1,p]$$

$$+[1,2,\ldots,r\mid 1,2,\ldots,r-2,r,p]$$

$$+[1,2,\ldots,r\mid 1,2,\ldots,r-1,p+1]$$

$$\therefore\ \frac{[1,2,\ldots,r\mid 1,2,\ldots,r-1,p]'}{[1,2,\ldots,r][1,2,\ldots,r-1]}$$

$$=\frac{[1,2,\ldots,r-1,r+1\mid 1,2,\ldots,r-1,p]}{[1,2,\ldots,r][1,2,\ldots,r-1]}$$

$$+\frac{[1,2,\ldots,r\mid 1,2,\ldots,r-2,r,p]}{[1,2,\ldots,r][1,2,\ldots,r-1]}+\Delta_r\,(p+1),$$

after which the proof is similar to proof of Formula E.

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* For a direct proof of this formula see *Bulletin, C.M.S.*, Vol. I pp. 240-42.

PARAMETRIC COEFFICIENTS IN THE DIFFERENTIAL GEOMETRY OF CURVES

BY

SYAMADAS MUKHOPADHYAYA.

IV. EXPRESSION OF THE CO-ORDINATES OF A POINT ON A CURVE IN AN V-SPACE AS POWER SERIES IN s.*

1. Let $\xi_1, \xi_2, .., \xi_n$ be the n co-ordinates of a point P on a curve in an n-space referred to principal axes at any point O on the curve. Then on the hypothesis of the existence of $\xi_1, \xi_2, .., \xi_n$ and their derivates of any order with respect to s, the arc length from O to P, in the domain (OP), we can show that

$$\xi_r = \underset{r}{\Delta}(r) . \frac{s^r}{r!} + \underset{r}{\Delta}(r+1) . \frac{s^{r+1}}{(r+1)!} + . + \underset{r}{\Delta}(m) \frac{s^m}{m!} + \text{etc.}$$

$$\text{for } r=1, 2, 3, .., n, \qquad (i)$$

$\Delta_r(m)$ having the same meaning as defined in Paper III.†

For if $X_1, X_2, .., X_n$ be the n co-ordinates of P referred to any system of orthogonal axes and $x_1, x_2, .., x_n$ the co-ordinates of O referred to the same system, then the direction cosines of the n principal axes at O are

$$\Delta_r(x_1), \Delta_r(x_2), .., \Delta_r(x), (r=1, 2, .., n)$$

* From *Bulletin, C. M. S.*, Vol. 3, 1911.

† $\Delta_r(m)$ stands for $[1, 2, ., r \mid 1, 2, .., r-1, m]$ divided by $[1, 2, .., r][1, 2, .., r-1]$ and $[1, 2, .., r \mid 1, 2, .., r-1, m]$ stands

$$\text{for} \begin{vmatrix} (1, 1), (1, 2), .., (1, r-1), (1, m) \\ (2, 1), (2, 2), .., (2, r-1), (2, m) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (r, 1), (r, 2) \;..\; (r, -r1), \;(r, m) \end{vmatrix}$$

$[1, 2, .., r]^2$ stands for $[1, 2, .., r \mid 1, 2, .., r]$,

$[1, 2, .., r-1]^2$ for $[1, 2, .., r-1 \mid 1, 2, .., r-1]$,

and (l, m) for $\sum x_p^{(l)} x_p^{(m)}, p=1, 2, .., n$.

When $r=1$, $\Delta_r(m)$ is equal to $(1, m) \div (1, 1)^{\frac{1}{2}}$.

where $\Delta_r\ (x_m)$ has the same meaning as defined in Paper III.*
Hence $\xi_r = \sum (X_p - x_p)\ \Delta_r\ (x_p)$. $(p=1, 2, ., n)$

$$= \sum \left(x_p' \cdot s + x_p'' \cdot \frac{s^2}{2!} + . + x^{(m)} \cdot \frac{s^m}{m!} + \text{etc.}\right) \Delta_r\ (x_p)$$

$$= s \sum x'_p\ \Delta_r\ (x_p) + \frac{s^2}{2!} \sum x''_p\ \Delta_r\ (x_p) + . . + \frac{s^m}{m!} \sum x_p^{(m)}\ \Delta_r\ (x_p) + \text{etc.}$$

$$= s\Delta_r\ (1) + \frac{s^2}{2!} \Delta_r\ (2) + \ . + \frac{s^m}{m!}\ \Delta_r\ (m) + \text{etc.}$$

by identity (a) paper III.

$$= \frac{s_r}{r!} \Delta_r\ (r) + \frac{s^{r+1}}{(r+1)!} \Delta_r\ (r+1) + . + \frac{s^m}{m!} \Delta_r\ (m) + \text{etc.}$$

as $\Delta_r\ (1)$, $\Delta_r\ (2)$, . , $\Delta_r\ (r-1)$ all vanish from definition of Δ_r.

Equation (i), written in full, gives us

$$\xi_1 = \Delta_1\ (1) \cdot s + \Delta_1\ (2) \cdot \frac{s^2}{2!} + . + \Delta_1\ (m) \cdot \frac{s^m}{m!} + \text{etc.}$$

$$\xi_2 = \Delta_2\ (2) \cdot \frac{s^2}{2!} + \Delta_2\ (3) \cdot \frac{s^3}{3!} + . + \Delta_2\ (m) \cdot \frac{s^m}{m!} + \text{etc.}$$

……………………………………………………………………………………

$$\xi_r = \Delta_r\ (r) \cdot \frac{s^r}{r!} + \Delta_r\ (r+1) \cdot \frac{s^{r+1}}{(r+1)!} + . + \Delta_r\ (m) \cdot \frac{s^m}{m!} + \text{etc.}$$

……………………………………………………………………………………

$$\xi_n = \Delta_n\ (n) \cdot \frac{s^n}{n!} + \Delta_n\ (n+1) \cdot \frac{s^{n+1}}{(n+1)!} + \Delta_n\ (m) \cdot \frac{s^m}{m!} + \text{etc.} \quad (ii)$$

We have now to express the coefficients of $\xi_1, \xi_2, \cdot, \xi_n$ in terms of the $n-1$ principal curvatures at O, $u_1, u_2, \cdot, u_{n-1}$ and their derivatives with respect to s.

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* $\Delta_r\ (x_m)$ stands for $[1, 2, .., r \mid 1, 2, , . r-1, x_m]$ divided by $[1, 2, .., r][1, 2, .., r-1]$ and

$[1, 2, . . , r \mid 1, 2, . . , r-1, x_m]$ stands for
$$\begin{vmatrix} (1, 1), & (1, 2), . . , & (1, r-1), & x'_m \\ (2, 1), & (2, 2), . . , & (2, r-1), & x''_m \\ \cdots & \cdots & \cdots & \cdots \\ (r, 1), & (r, 2), . . , & (r, r-2), & x_m^{(r)} \end{vmatrix}$$

We have from the definitions of the principal curvatures

$\Delta_2\ (2)=u_1,\ \Delta_3\ (3)=u_1\ u_2,\ \Delta_4\ (4)=u_1\ u_2\ u_3,\ ..$

$\Delta_r\ (r)=u_1\ u_2\ ..\ u_{r-1},\ ..\ \Delta_n\ (n)=u_1\ u_2,\ ..\ u_{n-1}.$

so that we can at once write down the first coefficient in each of the expressions (*ii*).

Evidently $\Delta_1\ (1)=(1,\ 1)^{\frac{1}{2}}$

$$=\left\{\left(\frac{dx_1}{ds}\right)^2+\left(\frac{dx_2}{ds}\right)^2+.+\left(\frac{dx_n}{ds}\right)^2\right\}^{\frac{1}{2}}=1$$

The second coefficients in (*ii*) are obtained from the equations

$\Delta_1\ (2)=0$

$\Delta_2\ (3)=\Delta'_2\ (2)+u_1\ \Delta_1\ (2)$

$\Delta_3\ (4)=\Delta'_3\ (3)+u_2\ \Delta_2\ (3)$

……

$\Delta_r\ (r+1)=\Delta'_r\ (r)+u_{r-1}\ \Delta_{r-1}\ (r)$

……

$$\Delta^n\ (n+1)=\Delta'_n\ (n)+u_{n-1}\ \Delta_{n-1}\ (n) \qquad (iii)$$

The first is evident since $\Delta_1\ (2)=(1,\ 2)\div(1,\ 1)^{\frac{1}{2}}$

$=(1,\ 2)=\frac{1}{2}\ (1,1)'=0.$ The rest follow from the general formula (E_1), of paper III, *viz.*,

$\Delta'_r\ (p)=\Delta_{r+1}\ (p)\ .\ u_r+\Delta_r\ (p+1)-\Delta_{r-1}\ (p)\ .\ u_{r-1},$

if we put r for p and observe that $\Delta_{r+1}\ (r)=0.$

Similarly, the third coefficients in (*ii*) are obtained from

$\Delta_1\ (3)=\Delta'_1\ (2)-u_1\ \Delta_2\ (2)$

$\Delta_2\ (4)=\Delta'_2\ (3)+u_1\ \Delta_1\ (3)-u_2\ \Delta_3\ (3)$

……

$\Delta_{n-1}\ (n+1)=\Delta'_{n-1}\ (n)+u_{n-2}\ \Delta_{n-2}\ (n)-u_{n-1}\ \Delta_n\ (n)$

$$\Delta_n\ (n+2)=\Delta'_n\ (n+1)+u_{n-1}\ \Delta_{n-1}\ (n+1) \qquad (iv)$$

and the fourth coefficients in (*ii*) are obtained from

$$\Delta_1\,(4)=\Delta'_1\,(3)-u_1\,\Delta_2\,(3)$$

$$\Delta_2\,(5)=\Delta'_2\,(4)+u_1\,\Delta_1\,(4)-u_2\,\Delta_3\,(4)$$

……………………………………………………

$$\Delta_{n-1}\,(n+2)=\Delta'_{n-1}\,(n+1)+u_{n-2}\,\Delta_{n-2}\,(n+1)-u_{n-1}\,\Delta_n\,(n+1)$$

$$\Delta_n\,(n+3)=\Delta'_n\,(n+2)+u_{n-1}\,\Delta_{n-1}\,(n+2) \qquad (v)$$

The above sets of equations (*iv*) and (*v*) are deduced from general formula (E_1) by substituting suitable values for p and r.

By continuing the above methods the coefficients of the expansions of $\xi_1, \xi_2, .. \xi_n$ in powers of s can be calculated with facility as far as we desire. G. B. Mathews in the Quarterly Journal of Mathematics, Vol. XXVI (1893), pp. 27-30, has obtained by a different kind of analysis similar formulae for curves in three-dimensional space.

The following tables which have been constructed by the above methods give the coefficients up to those of the *ninth* power of s. For convenience of printing $\alpha, \beta, \gamma, \delta, \epsilon, \theta, \phi$, etc., have been substituted for the principal curvatures $u_1, u_2, u_3, u_4, u_5, u_6$, etc.

$$\xi_1 = S - a^2\frac{S^3}{3!} - 3aa'\frac{S^4}{4!} + (a^4 - 4aa'' - 3a'^2 + a^2\beta^2)\frac{S^5}{5!} +$$

$S^6/6!$

$+10a^3a'$
$-5aa'''$
$-10a'a''$

$+5a^2\beta\beta'$
$+5aa'\beta^2$

$S^7/7!$

$-a^6$
$+20a^3a''$
$+45a^2a'^2$
$-6aa^{\mathrm{iv}}$
$-15a'a'''$
$-10a''^2$

$-2a^4\beta^2$
$-a^2\beta^4$
$+8a^2\beta'^2$
$+9a^2\beta\beta''$
$+32aa'\beta\beta'$
$+11aa''\beta^2$
$+5a'^2\beta^2$

$-a^2\beta^2\gamma^2$

$S^8/8!$

$-21a^5a'$
$+35a^3a'''$
$+210a^2a'a''$
$+105aa'^3$
$-7aa^{\mathrm{v}}$
$-21a'a^{\mathrm{iv}}$
$-35a''a'''$
$-14a^4\beta\beta'$
$-28a^3a'\beta^2$
$-14a^2\beta^3\beta'$
$+14a^2\beta\beta'''$
$+35a^2\beta'\beta''$
$-7aa'\beta^4$
$+63aa'\beta'^4$
$+70aa'\beta\beta''$
$+84aa''\beta\beta'$
$+21aa'''\beta^2$
$+42a'^2\beta\beta'$
$+21a'a''\beta^2$

$-7a^2\beta^2\gamma\gamma'$
$-7a^2\beta\beta'\gamma^2$
$-7aa'\beta^2\gamma^2$

$S^9/9!$

$+a^8$
$-56a^5a''$
$-210a^4a'^2$
$+56a^3a^{\mathrm{iv}}$
$+280a^2a''^2$
$+420a^2a'a'''$
$+840aa'^2a''$
$-8aa^{\mathrm{vi}}$
$+105a'^4$
$-28a'a^{\mathrm{v}}$
$-56a''a^{\mathrm{iv}}$
$-35a'''^2$

$+3a^6\beta^2$
$+3a^4\beta^4$
$-34a^4\beta\beta''$
$-32a^4\beta'^2$
$-226a^3a'\beta\beta'$
$-78a^3a''\beta^2$
$-162a^2a'^2\beta^2$
$+a^2\beta^6$
$-34a^2\beta^3\beta''$
$-87a^2\beta^2\beta'^2$
$+20a^2\beta\beta^{\mathrm{iv}}$
$+64a^2\beta'\beta'''$
$+45a^2\beta''^2$
$-116aa'\beta^3\beta'$
$+128aa'\beta\beta'''$
$+326aa'\beta'\beta''$
$-22aa''\beta^4$
$+214aa''\beta\beta''$
$+192aa''\beta'^2$
$+186aa'''\beta\beta'$
$+36aa^{\mathrm{iv}}\beta^2$
$-7a'^2\beta^4$
$+112a'^2\beta\beta''$
$+105a'^2\beta'^2$
$+210a'a''\beta\beta'$
$+42a'a'''\beta^2$
$+21a''^2\beta^2$

$+2a^4\beta^2\gamma^2$
$+2a^2\beta^4\gamma^2$
$+a^2\beta^2\gamma^4$
$-16a^2\beta^2\gamma\gamma''$
$-15a^2\beta^2\gamma'^2$
$-60a^2\beta\beta'\gamma\gamma'$
$-18a^2\beta\beta''\gamma^2$
$-12a^2\beta'^2\gamma^2$
$-58aa'\beta^2\gamma\gamma'$
$-58aa'\beta\beta'\gamma^2$
$-22aa''\beta^2\gamma^2$
$-7a'^2\beta^2\gamma^2$

$+a^2\beta^2\gamma^2\delta^2$

$$\xi_2 = a\frac{S^2}{2!} + a'\frac{S^3}{3!} + (-a^3 + a'' - a\beta^2)\frac{S^4}{4!} + (-6a^2a' + a''' - 3a\beta\beta' - 3a'\beta^2)\frac{S^2}{5!} +$$

$S^6/6!$

$+a^5$
$-10a^2a''$
$-15aa'^2$
$+a^{\mathrm{iv}}$

$+2a^3\beta^2$
$+a\beta^4$
$-3a\beta'^2$
$-4a\beta\beta''$
$-12a'\beta\beta'$
$-6a''\beta^2$
$+a\beta^2\gamma^2$

$S^7/7!$

$+15a^4a'$
$-15a^2a'''$
$-60aa'a''$
$-15a'^3$
$+a^{\mathrm{v}}$

$+10a^3\beta\beta'$
$+20a^2a'\beta^2$
$+10a\beta^3\beta'$
$-5a\beta\beta'''$
$-10a\beta'\beta''$
$+5a'\beta^4$
$-20a'\beta\beta''$
$-15a'\beta'^2$
$-30a''\beta\beta'$
$-10a'''\beta^2$

$+5a\beta^2\gamma\gamma'$
$+5a\beta\beta'\gamma^2$
$+5a'\beta^2\gamma^2$

$S^8/8!$

$-a^7$
$+35a^4a''$
$+105a^3a'^2$
$-21a^2a^{\mathrm{iv}}$
$-105aa'a'''$
$-70aa''^2$
$-105a'^2a''$
$+a^{\mathrm{vi}}$

$-3a^5\beta^2$
$-3a^3\beta^4$
$+18a^3\beta'^2$
$+20a^3\beta\beta''$
$+114a^2a'\beta\beta'$
$+50a^2a''\beta^2$
$-a\beta^6$
$+20a\beta^3\beta''$
$+45a\beta^2\beta'^2$
$+78aa'^2\beta^2$
$-6a\beta\beta^{\mathrm{iv}}$
$-15a\beta'\beta'''$
$-10a\beta''^2$
$+60a'\beta^3\beta'$
$-30a'\beta\beta'''$
$-60a'\beta'\beta''$
$+15a''\beta^4$
$-45a''\beta'^2$
$-60a''\beta\beta''$
$-60a'''\beta\beta'$
$-15a^{\mathrm{iv}}\beta^2$
$-2a^3\beta^2\gamma^2$
$-2a\beta^4\gamma^2$
$+8a\beta^2\gamma'^2$
$-a\beta^2\gamma^4$
$+9a\beta^2\gamma\gamma''$
$+32a\beta\beta'\gamma\gamma'$
$+11a\beta\beta''\gamma^2$
$+5a\beta'^2\gamma^2$
$+30a'\beta^2\gamma\gamma'$
$+30a'\beta\beta'\gamma^2$
$+15a''\beta^2\gamma^2$

$-a\beta^2\gamma^2\delta^2$

$S^9/9!$

$-28a^6a'$
$+70a^4a'''$
$+560a^3a'a''$
$+420a^2a'^3$
$-28a^2a^{\mathrm{v}}$
$-168aa'a^{\mathrm{iv}}$
$-280aa''a'''$
$-210a'^2a'''$

$-280\alpha'\alpha''^2$
$+\alpha^{vii}$

$-21\alpha^5\beta\beta'$
$-63\alpha^4\alpha'\beta^2$
$-42\alpha^3\beta^3\beta'$
$+35\alpha^3\beta\beta'''$
$+91\alpha^3\beta'\beta''$
$-42\alpha^2\alpha'\beta^4$
$+259\alpha^2\alpha'\beta\beta''$
$+231\alpha^2\alpha'\beta'^2$
$+329\alpha^2\alpha''\beta\beta'$
$+105\alpha^2\alpha'''\beta^2$

$-21\alpha\beta^5\beta'$
$+35\alpha\beta^3\beta'''$
$+210\alpha\beta^2\beta'\beta''$
$+483\alpha\alpha'^2\beta\beta'$
$+105\alpha\beta\beta'^3$
$-7\alpha\beta\beta^{v}$
$-21\alpha\beta'\beta^{iv}$
$-35\alpha\beta''\beta'''$
$+441\alpha\alpha'\alpha''\beta^2$
$+126\alpha'^3\beta^2$
$-7\alpha'\beta^6$
$+140\alpha'\beta^3\beta''$
$+315\alpha'\beta^2\beta'^2$
$-42\alpha'\beta\beta^{iv}$

$-105\alpha'\beta'\beta'''$
$-70\alpha'\beta''^2$
$+210\alpha''\beta^3\beta'$
$-105\alpha''\beta\beta'''$
$-210\alpha''\beta'\beta''$
$+35\alpha'''\beta^4$
$-140\alpha'''\beta\beta''$
$-105\alpha'''\beta'^2$
$-105\alpha^{iv}\beta\beta'$
$-21\alpha^{v}\beta^2$

$-14\alpha^3\beta^2\gamma\gamma'$
$-14\alpha^3\beta\beta'\gamma^2$
$-28\alpha^2\alpha'\beta^2\gamma^2$

$-14\alpha\beta^4\gamma\gamma'$
$-28\alpha\beta^3\beta'\gamma^2$
$-14\alpha\beta^2\gamma^3\gamma'$
$+14\alpha\beta^2\gamma\gamma'''$
$+35\alpha\beta^2\gamma'\gamma''$
$-7\alpha\beta\beta'\gamma^4$
$+70\alpha\beta\beta'\gamma\gamma''$
$+63\alpha\beta\beta'\gamma'^2$
$+84\alpha\beta\beta''\gamma\gamma'$
$+21\alpha\beta\beta'''\gamma^2$
$+42\alpha\beta'^2\gamma\gamma'$
$+21\alpha\beta'\beta''\gamma^2$
$-14\alpha'\beta^4\gamma^2$
$-7\alpha'\beta^2\gamma^4$

$+63\alpha'\beta^2\gamma\gamma''$
$+56\alpha'\beta^2\gamma'^2$
$+224\alpha'\beta\beta'\gamma\gamma'$
$+77\alpha'\beta\beta''\gamma^2$
$+35\alpha'\beta'^2\gamma^2$
$+105\alpha''\beta^2\gamma\gamma'$
$+105\alpha''\beta\beta'\gamma^2$
$+35\alpha'''\beta^2\gamma^2$

$-7\alpha\beta^2\gamma^2\delta\delta'$
$-7\alpha\beta^2\gamma\gamma'\delta^2$
$-7\alpha\beta\beta'\gamma^2\delta^2$
$-7\alpha'\beta\gamma^2\delta^2$

$$\xi_3=\alpha\beta\frac{S^3}{3!}+(\alpha\beta'+2'\alpha\beta)\frac{S^4}{4!}+(-\alpha^3\beta-\alpha\beta^3+\alpha\beta''+3\alpha'\beta'+3\alpha''\beta-\alpha\beta\gamma^2)\frac{S^5}{5!}+$$

$S^6/6!$

$-\alpha^3\beta'$
$+\alpha\beta'''$
$-4\alpha'\beta^3$
$+4\alpha'\beta''$
$+6\alpha''\beta'$
$+4\alpha'''\beta$
$-9\alpha^2\alpha'\beta$
$+6\alpha\beta^2\beta'$

$-3\alpha\beta'\gamma^2$
$-4\alpha'\beta\gamma^2$
$-3\alpha\beta\gamma\gamma'$

$S^7/7!$

$+\alpha^5\beta$
$+2\alpha^3\beta^3$
$-\alpha^3\beta''$
$-12\alpha^2\alpha'\beta'$
$+\alpha\beta^5$
$-19\alpha^2\alpha''\beta$
$-10\alpha\beta^2\beta''$
$-15\alpha\beta\beta'^2$
$-33\alpha\alpha'^2\beta$
$+\alpha\beta^{iv}$

$-30\alpha'\beta^2\beta'$
$+5\alpha'\beta'''$
$-10\alpha''\beta^3$
$+10\alpha''\beta''$
$+10\alpha'''\beta'$
$+5\alpha^{iv}\beta$

$+\alpha^3\beta\gamma^2$
$+2\alpha\beta^3\gamma^2$
$+\alpha\beta\gamma^4$
$-4\alpha\beta\gamma\gamma''$
$-3\alpha\beta\gamma'^2$
$-12\alpha\beta'\gamma\gamma'$
$-6\alpha\beta''\gamma^2$
$-15\alpha'\beta\gamma\gamma'$
$-15\alpha'\beta'\gamma^2$
$-10\alpha''\beta\gamma^2$

$+\alpha\beta\gamma^2\delta^2$

$S^8/8!$

$+\alpha^5\beta'$
$+20\alpha^4\alpha'\beta$
$+16\alpha^3\beta^2\beta'$
$+20\alpha^2\alpha'\beta^3$

$-\alpha^3\beta'''$
$-15\alpha^2\alpha'\beta''$
$-31\alpha^2\alpha''\beta'$
$-34\alpha^2\alpha'''\beta$
$+15\alpha\beta^4\beta'$
$-15\alpha\beta^2\beta'''$
$-57\alpha\alpha'^2\beta'$
$-164\alpha\alpha'\alpha''\beta$
$-60\alpha\beta\beta'\beta''$
$-15\alpha\beta'^3$
$+\alpha\beta^{v}$
$-48\alpha'^3\beta$
$+6\alpha'\beta^5$

$-60\alpha'\beta^2\beta''$
$-90\alpha'\beta\beta'^2$

$+6\alpha'\beta^{iv}$
$-90\alpha''\beta^2\beta'$
$+15\alpha''\beta'''$
$-20\alpha'''\beta^3$
$+20\alpha'''\beta''$
$+15\alpha^{iv}\beta'$
$+6\alpha^{v}\beta$

$+3\alpha^3\beta\gamma\gamma'$
$+3\alpha^3\beta'\gamma^2$

$+15\alpha^2\alpha'\beta\gamma^2$
$+10\alpha\beta^3\gamma\gamma'$

$+20\alpha\beta^2\beta'\gamma^2$
$+10\alpha\beta\gamma^3\gamma'$
$-5\alpha\beta\gamma\gamma'''$
$-10\alpha\beta\gamma'\gamma''$
$+5\alpha\beta'\gamma^4$
$-20\alpha\beta'\gamma\gamma''$
$-15\alpha\beta'\gamma'^2$
$-30\alpha\beta''\gamma\gamma'$
$-10\alpha\beta'''\gamma^2$
$+6\alpha'\beta\gamma^4$
$+12\alpha'\beta^3\gamma^2$

$-24\alpha'\beta\gamma\gamma''$
$-18\alpha'\beta\gamma'^2$
$-72\alpha'\beta'\gamma\gamma'$
$-36\alpha'\beta''\gamma^2$
$-45\alpha''\beta\gamma\gamma'$
$-45\alpha''\beta'\gamma^2$
$-20\alpha'''\beta\gamma^2$

$+5\alpha\beta\gamma^2\delta\delta'$
$+5\alpha\beta\gamma\gamma'\delta^2$
$+5\alpha\beta'\gamma^2\delta^2$
$+6\alpha'\beta\gamma^2\delta^2$

$S^9/9!$

$-\alpha^7\beta$
$-3\alpha^5\beta^3$
$+\alpha^5\beta''$
$+25\alpha^4\alpha'\beta'$
$+55\alpha^4\alpha''\beta$
$-3\alpha^3\beta^5$
$+185\alpha^3\alpha'^2\beta$
$+36\alpha^3\beta^2\beta''$
$+50\alpha^3\beta\beta'^2$
$-\alpha^3\beta^{iv}$
$+240\alpha^2\alpha'\beta^2\beta'$
$-18\alpha^2\alpha'\beta'''$
$+76\alpha^2\alpha''\beta^3$
$-46\alpha^2\alpha''\beta''$
$-65\alpha^2\alpha'''\beta'$
$-55\alpha^2\alpha^{iv}\beta$
$+130\alpha\alpha'^2\beta^3$
$-87\alpha\alpha'^2\beta''$
$-340\alpha\alpha'\alpha''\beta'$
$-337\alpha\alpha'\alpha'''\beta$
$-\alpha\beta^7$
$+35\alpha\beta^4\beta''$
$+105\alpha\beta^3\beta'^2$

$-21\alpha\beta^2\beta^{iv}$
$-105\alpha\beta\beta'\beta'''$
$-70\alpha\beta\beta''^2$
$-105\alpha\beta'^2\beta''$
$-234\alpha\alpha''^2\beta$
$+\alpha\beta^{vi}$
$-105\alpha'^3\beta'$
$-413\alpha'^2\alpha''\beta$
$+105\alpha'\beta^4\beta'$
$-105\alpha'\beta^2\beta'''$
$-420\alpha'\beta\beta'\beta''$
$-105\alpha'\beta'^3$
$+7\alpha'\beta^{v}$
$+21\alpha''\beta^5$
$-210\alpha''\beta^2\beta''$
$-315\alpha''\beta\beta'^2$
$+21\alpha''\beta^{iv}$
$-210\alpha'''\beta^2\beta'$
$+35\alpha'''\beta'''$

$-35\alpha^{iv}\beta^3$
$+35\alpha^{iv}\beta''$
$+21\alpha^{v}\beta'$
$+7\alpha^{vi}\beta$

$-\alpha^5\beta\gamma^2$
$-4\alpha^3\beta^3\gamma^2$
$-\alpha^3\beta\gamma^4$
$+4\alpha^3\beta\gamma\gamma''$
$+3\alpha^3\beta\gamma'^2$
$+12\alpha^3\beta'\gamma\gamma'$
$+6\alpha^3\beta''\gamma^2$
$+54\alpha^2\alpha'\beta\gamma\gamma'$
$+54\alpha^2\alpha'\beta'\gamma^2$
$+46\alpha^2\alpha''\beta\gamma^2$
$-3\alpha\beta^5\gamma^2$
$-3\alpha\beta^3\gamma^4$
$+20\alpha\beta^3\gamma\gamma''$

$18\alpha\beta^3\gamma'^2$
$+114\alpha\beta^2\beta'\gamma\gamma'$
$+50\alpha\beta^2\beta''\gamma^2$
$-\alpha\beta\gamma^6$
$+20\alpha\beta\gamma^3\gamma''$
$+45\alpha\beta\gamma^2\gamma'^2$
$-6\alpha\beta\gamma\gamma^{iv}$
$+78\alpha\beta\beta'^2\gamma^2$
$-15\alpha\beta\gamma'\gamma'''$
$-10\alpha\beta\gamma''^2$
$+87\alpha\alpha'^2\beta\gamma^2$
$+60\alpha\beta'\gamma^3\gamma'$
$-30\alpha\beta'\gamma\gamma'''$
$-60\alpha\beta'\gamma'\gamma''$
$+15\alpha\beta''\gamma^4$
$-60\alpha\beta''\gamma\gamma''$
$-45\alpha\beta''\gamma'^2$
$-60\alpha\beta'''\gamma\gamma'$
$-15\alpha\beta^{iv}\gamma^2$

$+70\alpha'\beta^3\gamma\gamma'$
$+140\alpha'\beta^2\beta'\gamma^2$
$+70\alpha'\beta\gamma^3\gamma'$
$-35\alpha'\beta\gamma\gamma'''$
$-70\alpha'\beta\gamma'\gamma''$
$+35\alpha'\beta'\gamma^4$
$-140\alpha'\beta'\gamma\gamma''$
$-105\alpha'\beta'\gamma'^2$
$-210\alpha'\beta''\gamma\gamma''$
$-70\alpha'\beta'''\gamma^2$
$+42\alpha''\beta^3\gamma^2$
$+21\alpha''\beta\gamma^4$
$-84\alpha''\beta\gamma\gamma''$
$-63\alpha''\beta\gamma'^2$
$-252\alpha''\beta'\gamma\gamma'$
$-126\alpha''\beta''\gamma^2$
$-105\alpha'''\beta\gamma\gamma'$
$-105\alpha'''\beta'\gamma^2$
$-35\alpha^{iv}\beta\gamma^2$

$-\alpha^3\beta\gamma^2\delta^2$
$-2\alpha\beta^3\gamma^2\delta^2$
$-2\alpha\beta\gamma^4\delta^2$
$-\alpha\beta\gamma^2\delta^4$
$+9\alpha\beta\gamma^2\delta\delta''$
$+8\alpha\beta\gamma^2\delta'^2$
$+32\alpha\beta\gamma\gamma'\delta\delta'$
$+11\alpha\beta\gamma\gamma''\delta^2$
$+5\alpha\beta\gamma'^2\delta^2$
$+30\alpha\beta'\gamma^2\delta\delta'$
$+30\alpha\beta'\gamma\gamma'\delta^2$
$+15\alpha\beta''\gamma^2\delta^2$
$+35\alpha'\beta\gamma^2\delta\delta'$
$+35\alpha'\beta\gamma\gamma'\delta^2$
$+35\alpha'\beta'\gamma^2\delta^2$
$+21\alpha''\beta\gamma^2\delta^2$

$-\alpha\beta\gamma^2\delta^2\epsilon^2$

$$\xi_4 = \alpha\beta\gamma\,\frac{S^4}{4!} + (\alpha\beta\gamma' + 2\alpha\beta'\gamma + 3\alpha'\beta\gamma)\,\frac{S^5}{5!} + $$

$S^6/6!$

$-\alpha^3\beta\gamma$
$-\alpha\beta^3\gamma$
$-\alpha\beta\gamma^3$
$+\alpha\beta\gamma''$
$+3\alpha\beta'\gamma'$
$+3\alpha\beta''\gamma$
$+4\alpha'\beta\gamma'$
$+8\alpha'\beta'\gamma$
$+6\alpha''\beta\gamma$
$-\alpha\beta\gamma\delta^2$

$S^7/7!$

$\alpha^3\beta\gamma'$
$-2\alpha^3\beta'\gamma$
$-12\alpha^2\alpha'\beta\gamma$
$-\alpha\beta^3\gamma'$
$-9\alpha\beta^2\beta'\gamma$
$-6\alpha\beta\gamma^2\gamma'$
$+\alpha\beta\gamma'''$

$-4\alpha\beta'\gamma^3$
$+4\alpha\beta'\gamma''$
$+6\alpha\beta''\gamma'$
$+4\alpha\beta'''\gamma$
$-5\alpha'\beta^3\gamma$
$-5\alpha'\beta\gamma^3$
$+5\alpha'\beta\gamma''$
$+15\alpha'\beta'\gamma'$
$+15\alpha'\beta''\gamma$
$+10\alpha''\beta\gamma'$
$+20\alpha''\beta'\gamma$
$+10\alpha'''\beta\gamma$
$-3\alpha\beta\gamma'\delta^2$
$-4\alpha\beta'\gamma\delta^2$
$-3\alpha\beta\gamma\delta\delta'$
$-5\alpha'\beta\gamma\delta^2$

$S^8/8!$

$+\alpha^5\beta\gamma$
$+2\alpha^3\beta^3\gamma$
$+\alpha^3\beta\gamma^3$
$-\alpha^3\beta\gamma''$

$-3\alpha^3\beta'\gamma'$
$-8\alpha^3\beta''\gamma$
$-15\alpha^2\alpha'\beta\gamma'$
$-30\alpha^2\alpha'\beta'\gamma$
$-31\alpha^2\alpha''\beta\gamma$
$+\alpha\beta^5\gamma$
$+2\alpha\beta^3\gamma^3$
$-\alpha\beta^3\gamma''$
$-12\alpha\beta^2\beta'\gamma'$
$-19\alpha\beta^2\beta''\gamma$
$+\alpha\beta\gamma^5$
$-57\alpha\alpha'^2\beta\gamma$
$-33\alpha\beta\beta'^2\gamma$
$-10\alpha\beta\gamma^2\gamma''$
$-15\alpha\beta\gamma\gamma'^2$
$+\alpha\beta\gamma^{iv}$
$-30\alpha\beta'\gamma^2\gamma'$
$+5\alpha\beta'\gamma'''$
$-10\alpha\beta''\gamma^3$
$+10\alpha\beta''\gamma''$
$+10\alpha\beta'''\gamma'$
$+5\alpha\beta^{iv}\gamma$

$-6\alpha'\beta^3\gamma'$
$-54\alpha'\beta^2\beta'\gamma$
$-36\alpha'\beta\gamma^2\gamma'$
$+6\alpha'\beta\gamma'''$
$-24\alpha'\beta'\gamma^3$
$+24\alpha'\beta'\gamma''$
$+36\alpha'\beta''\gamma'$
$+24\alpha'\beta'''\gamma$
$-15\alpha''\beta^3\gamma$
$-15\alpha''\beta\gamma^3$
$+15\alpha''\beta\gamma''$
$+45\alpha''\beta'\gamma'$
$+45\alpha''\beta''\gamma$
$+20\alpha'''\beta\gamma'$
$+40\alpha'''\beta'\gamma$
$+15\alpha^{iv}\beta\gamma$

$+\alpha^3\beta\gamma\delta^2$
$+\alpha\beta^3\gamma\delta^2$
$+2\alpha\beta\gamma^3\delta^2$
$+\alpha\beta\gamma\delta^4$
$-3\alpha\beta\gamma\delta'^2$

$-4\alpha\beta\gamma\delta\delta''$
$-12\alpha\beta\gamma'\delta\delta'$
$-6\alpha\beta\gamma''\delta^2$
$-15\alpha\beta'\gamma\delta\delta'$
$-15\alpha\beta'\gamma'\delta^2$
$-10\alpha\beta''\gamma\delta^2$
$-18\alpha'\beta\gamma\delta\delta'$
$-18\alpha'\beta\gamma'\delta^2$
$-24\alpha'\beta'\gamma\delta^2$
$-15\alpha''\beta\gamma\delta^2$

$+\alpha\beta\gamma\delta^2\epsilon^2$

$S^9/9!$

$+\alpha^5\beta\gamma'$
$+2\alpha^5\beta'\gamma$
$+25\alpha^4\alpha'\beta\gamma$
$+2\alpha^3\beta^3\gamma'$
$+22\alpha^3\beta^2\beta'\gamma$
$+6\alpha^3\beta\gamma^2\gamma'$
$-\alpha^3\beta\gamma'''$
$+4\alpha^3\beta'\gamma^3$

$$\begin{gathered}
-4\alpha^3\beta'\gamma'' \\
-6\alpha^3\beta''\gamma' \\
-4\alpha^3\beta'''\gamma \\
+32\alpha^2\alpha'\beta^3\gamma \\
+18\alpha^2\alpha'\beta\gamma^3 \\
-18\alpha^2\alpha'\beta\gamma'' \\
-54\alpha^2\alpha'\beta'\gamma' \\
-54\alpha^2\alpha'\beta''\gamma \\
-46\alpha^2\alpha''\beta\gamma' \\
-92\alpha^2\alpha''\beta'\gamma \\
-65\alpha^2\alpha'''\beta\gamma \\
-87\alpha\alpha'^2\beta\gamma' \\
-174\alpha\alpha'^2\beta'\gamma \\
-340\alpha\alpha'\alpha''\beta\gamma \\
+\alpha\beta^5\gamma' \\
+20\alpha\beta^4\beta'\gamma \\
+16\alpha\beta^3\gamma^2\gamma' \\
-\alpha\beta^3\gamma''' \\
+26\alpha\beta^2\beta'\gamma^3 \\
-15\alpha\beta^2\beta'\gamma'' \\
-31\alpha\beta^2\beta''\gamma' \\
-35\alpha\beta^2\beta'''\gamma \\
-57\alpha\beta\beta'^2\gamma' \\
-164\alpha\beta\beta'\beta''\gamma
\end{gathered}$$

$$\begin{gathered}
+15\alpha\beta\gamma^4\gamma' \\
-15\alpha\beta\gamma^2\gamma''' \\
-60\alpha\beta\gamma\gamma'\gamma'' \\
-15\alpha\beta\gamma'^3 \\
+\alpha\beta\gamma^{\text{v}} \\
-48\alpha\beta'^3\gamma \\
+6\alpha\beta'\gamma^5 \\
-60\alpha\beta'\gamma^2\gamma'' \\
-90\alpha\beta'\gamma\gamma'^2 \\
+6\alpha\beta'\gamma^{\text{iv}} \\
-90\alpha\beta''\gamma^2\gamma' \\
+15\alpha\beta''\gamma''' \\
-20\alpha\beta'''\gamma^3 \\
+20\alpha\beta'''\gamma'' \\
+15\alpha\beta^{\text{iv}}\gamma' \\
+6\alpha\beta^{\text{v}}\gamma \\
-105\alpha'^3\beta\gamma \\
+7\alpha'\beta^5\gamma \\
+14\alpha'\beta^3\gamma^3 \\
-7\alpha'\beta^3\gamma'' \\
-84\alpha'\beta^2\beta'\gamma' \\
-133\alpha'\beta^2\beta''\gamma \\
-231\alpha'\beta\beta'^2\gamma \\
+7\alpha'\beta\gamma^5
\end{gathered}$$

$$\begin{gathered}
-70\alpha'\beta\gamma^2\gamma'' \\
-105\alpha'\beta\gamma\gamma'^2 \\
+7\alpha'\beta\gamma^{\text{iv}} \\
-210\alpha'\beta'\gamma^2\gamma' \\
+35\alpha'\beta'\gamma''' \\
-60\alpha'\beta''\gamma^3 \\
+60\alpha'\beta''\gamma'' \\
+60\alpha'\beta'''\gamma' \\
+35\alpha'\beta^{\text{iv}}\gamma \\
-21\alpha''\beta^3\gamma' \\
-174\alpha''\beta^2\beta'\gamma \\
-126\alpha''\beta\gamma^2\gamma' \\
+21\alpha''\beta\gamma''' \\
-84\alpha''\beta'\gamma^3 \\
+84\alpha''\beta'\gamma'' \\
+126\alpha''\beta''\gamma' \\
+84\alpha''\beta'''\gamma \\
-21\alpha'''\beta^3\gamma \\
-35\alpha'''\beta\gamma^3 \\
+35\alpha'''\beta\gamma'' \\
+105\alpha'''\beta'\gamma' \\
+105\alpha'''\beta''\gamma \\
+35\alpha^{\text{iv}}\beta\gamma' \\
+70\alpha^{\text{iv}}\beta'\gamma
\end{gathered}$$

$$+21\alpha^{\text{v}}\beta\gamma$$

$$\begin{gathered}
+3\alpha^3\beta\gamma\delta\delta' \\
+3\alpha^3\beta\gamma'\delta^2 \\
+4\alpha^3\beta'\gamma\delta^2 \\
+18\alpha^2\alpha'\beta\gamma\delta^2 \\
+3\alpha\beta^3\gamma\delta\delta' \\
+3\alpha\beta^3\gamma'\delta^2 \\
+15\alpha\beta^2\beta'\gamma\delta^2 \\
+20\alpha\beta\gamma^3\delta\delta' \\
+20\alpha\beta\gamma^2\gamma'\delta^2 \\
+10\alpha\beta\gamma\delta^3\delta' \\
-10\alpha\beta\gamma\delta'\delta'' \\
-5\alpha\beta\gamma\delta\delta''' \\
+5\alpha\beta\gamma'\delta^4 \\
-15\alpha\beta\gamma'\delta'^2 \\
-20\alpha\beta\gamma'\delta\delta'' \\
-30\alpha\beta\gamma''\delta\delta' \\
-10\alpha\beta\gamma'''\delta^2 \\
+12\alpha\beta'\gamma^3\delta^2 \\
+6\alpha\beta'\gamma\delta^4 \\
-18\alpha\beta'\gamma\delta'^2 \\
-24\alpha\beta'\gamma\delta\delta'' \\
-72\alpha\beta'\gamma'\delta\delta'
\end{gathered}$$

$$\begin{gathered}
-36\alpha\beta'\gamma''\delta^2 \\
-45\alpha\beta''\gamma\delta\delta' \\
-45\alpha\beta''\gamma'\delta^2 \\
-20\alpha\beta'''\gamma\delta^2 \\
+7\alpha'\beta^3\gamma\delta^2 \\
+14\alpha'\beta\gamma^3\delta^2 \\
+7\alpha'\beta\gamma\delta^4 \\
-21\alpha'\beta\gamma\delta'^2 \\
-28\alpha'\beta\gamma\delta\delta'' \\
-84\alpha'\beta\gamma'\delta\delta' \\
-42\alpha'\beta\gamma''\delta^2 \\
-105\alpha'\beta'\gamma\delta\delta' \\
-105\alpha'\beta'\gamma'\delta^2 \\
-70\alpha'\beta''\gamma\delta^2 \\
-63\alpha''\beta\gamma\delta\delta' \\
-63\alpha''\beta\gamma'\delta^2 \\
-84\alpha''\beta'\gamma\delta^2 \\
-35\alpha'''\beta\gamma\delta^2
\end{gathered}$$

$$\begin{gathered}
+5\alpha\beta\gamma\delta^2\epsilon\epsilon' \\
+5\alpha\beta\gamma\delta\delta'\epsilon^2 \\
+5\alpha\beta\gamma'\delta^2\epsilon^2 \\
+6\alpha\beta'\gamma\delta^2\epsilon^2 \\
+7\alpha'\beta\gamma\delta^2\epsilon^2
\end{gathered}$$

$$\xi_5=\alpha\beta\gamma\delta\,\frac{\mathrm{S}^5}{5!}+(\alpha\beta\gamma\delta'+2\alpha\beta\gamma'\delta+3\alpha\beta'\gamma\delta+4\alpha'\beta\gamma\delta)\,\frac{\mathrm{S}^6}{6!}\;+$$

$\mathrm{S}^7/7!$

$$\begin{gathered}
-\alpha^3\beta\gamma\delta \\
-\alpha\beta^3\gamma\delta \\
-\alpha\beta\gamma^3\delta \\
-\alpha\beta\gamma\delta^3 \\
+\alpha\beta\gamma\delta'' \\
+3\alpha\beta\gamma''\delta \\
+3\alpha\beta\gamma'\delta' \\
+4\alpha\beta'\gamma\delta' \\
+8\alpha\beta'\gamma'\delta \\
+6\alpha\beta''\gamma\delta \\
+5\alpha'\beta\gamma\delta' \\
+10\alpha'\beta\gamma'\delta \\
+15\alpha'\beta'\gamma\delta \\
+10\alpha''\beta\gamma\delta
\end{gathered}$$

$$-\alpha\beta\gamma\delta\epsilon^2$$

$\mathrm{S}^8/8!$

$$\begin{gathered}
-\alpha^3\beta\gamma\delta' \\
-2\alpha^3\beta\gamma'\delta \\
-3\alpha^3\beta'\gamma\delta \\
-15\alpha^2\alpha'\beta\gamma\delta \\
-\alpha\beta^3\gamma\delta' \\
-2\alpha\beta^3\gamma'\delta \\
-12\alpha\beta^2\beta'\gamma\delta \\
-\alpha\beta\gamma^3\delta' \\
-9\alpha\beta\gamma^2\gamma'\delta \\
-6\alpha\beta\gamma\delta^2\delta' \\
+\alpha\beta\gamma\delta''' \\
-4\alpha\beta\gamma'\delta^3 \\
+4\alpha\beta\gamma'\delta'' \\
+6\alpha\beta\gamma''\delta'
\end{gathered}$$

$$\begin{gathered}
+4\alpha\beta\gamma'''\delta \\
-5\alpha\beta'\gamma^3\delta \\
-5\alpha\beta'\gamma\delta^3 \\
+5\alpha\beta'\gamma\delta'' \\
+15\alpha\beta'\gamma'\delta' \\
+15\alpha\beta'\gamma''\delta \\
+10\alpha\beta''\gamma\delta' \\
+20\alpha\beta''\gamma'\delta \\
+10\alpha\beta'''\gamma\delta \\
-6\alpha'\beta^3\gamma\delta \\
-6\alpha'\beta\gamma^3\delta \\
-6\alpha'\beta\gamma\delta^3 \\
+6\alpha'\beta\gamma\delta'' \\
+18\alpha'\beta\gamma'\delta' \\
+18\alpha'\beta\gamma''\delta \\
+24\alpha'\beta'\gamma\delta' \\
+48\alpha'\beta'\gamma'\delta
\end{gathered}$$

$$\begin{gathered}
+36\alpha'\beta''\gamma\delta \\
+15\alpha''\beta\gamma\delta' \\
+30\alpha''\beta\gamma'\delta \\
+45\alpha''\beta'\gamma\delta \\
+20\alpha'''\beta\gamma\delta
\end{gathered}$$

$$\begin{gathered}
-3\alpha\beta\gamma\delta'\epsilon^2 \\
-3\alpha\beta\gamma\delta\epsilon\epsilon' \\
-4\alpha\beta\gamma'\delta\epsilon^2 \\
-5\alpha\beta'\gamma\delta\epsilon^2 \\
-6\alpha'\beta\gamma\delta\epsilon^2
\end{gathered}$$

$\mathrm{S}^9/9!$

$$\begin{gathered}
+\alpha^5\beta\gamma\delta \\
+2\alpha^3\beta^3\gamma\delta \\
+\alpha^3\beta\gamma^3\delta \\
+\alpha^3\beta\gamma\delta^3
\end{gathered}$$

$$\begin{gathered}
+\alpha\beta^5\gamma\delta \\
+2\alpha\beta^3\gamma^3\delta \\
+\alpha\beta^3\gamma\delta^3 \\
+\alpha\beta\gamma^5\delta \\
+2\alpha\beta\gamma^3\delta^3 \\
+\alpha\beta\gamma\delta^5 \\
-\alpha^3\beta\gamma\delta'' \\
-3\alpha^3\beta\gamma'\delta' \\
-3\alpha^3\beta\gamma''\delta \\
-4\alpha^3\beta'\gamma\delta' \\
-8\alpha^3\beta'\gamma'\delta \\
-6\alpha^3\beta''\gamma\delta \\
-18\alpha^2\alpha'\beta\gamma\delta' \\
-36\alpha^2\alpha'\beta\gamma'\delta \\
-54\alpha^2\alpha'\beta'\gamma\delta \\
-46\alpha^2\alpha''\beta\gamma\delta \\
-87\alpha\alpha'^2\beta\gamma\delta
\end{gathered}$$

$-\alpha\beta^3\gamma\delta''$
$-3\alpha\beta^3\gamma'\delta'$
$-3\alpha\beta^3\gamma''\delta$
$-15\alpha\beta^2\beta'\gamma\delta'$
$-30\alpha\beta^2\beta'\gamma'\delta$
$-31\alpha\beta^2\beta''\gamma\delta$
$-57\alpha\beta\beta'^2\gamma\delta$
$-\alpha\beta\gamma^3\delta''$
$-12\alpha\beta\gamma^2\gamma'\delta'$
$-19\alpha\beta\gamma^2\gamma''\delta$
$-33\alpha\beta\gamma\gamma'^2\delta$
$-10\alpha\beta\gamma\delta^2\delta''$
$-15\alpha\beta\gamma\delta\delta'^2$
$+\alpha\beta\gamma\delta^{iv}$
$-30\alpha\beta\gamma'\delta^2\delta'$
$+5\alpha\beta\gamma'\delta'''$
$-10\alpha\beta\gamma''\delta^3$
$+10\alpha\beta\gamma''\delta''$
$+10\alpha\beta\gamma'''\delta'$

$+5\alpha\beta\gamma^{iv}\delta$
$-6\alpha\beta'\gamma^3\delta'$
$-54\alpha\beta'\gamma^2\gamma'\delta$
$-36\alpha\beta'\gamma\delta^2\delta'$
$+6\alpha\beta'\gamma\delta'''$
$-24\alpha\beta'\gamma'\delta^3$
$+24\alpha\beta'\gamma'\delta''$
$+36\alpha\beta'\gamma''\delta'$
$+24\alpha\beta'\gamma'''\delta$
$-15\alpha\beta''\gamma^3\delta$
$-15\alpha\beta''\gamma\delta^3$
$+15\alpha\beta''\gamma\delta''$
$+45\alpha\beta''\gamma'\delta'$
$+45\alpha\beta''\gamma''\delta$
$+20\alpha\beta'''\gamma\delta'$
$+40\alpha\beta'''\gamma'\delta$
$+15\alpha\beta^{iv}\gamma\delta$
$-7\alpha'\beta^3\gamma\delta'$
$-14\alpha'\beta^3\gamma'\delta$

$-84\alpha'\beta^2\beta'\gamma\delta$
$-7\alpha'\beta\gamma^3\delta'$
$-63\alpha'\beta\gamma^2\gamma'\delta$
$-42\alpha'\beta\gamma\delta^2\delta'$
$+7\alpha'\beta\gamma\delta'''$
$-28\alpha'\beta\gamma'\delta^3$
$+28\alpha'\beta\gamma'\delta''$
$+42\alpha'\beta\gamma''\delta'$
$+28\alpha'\beta\gamma'''\delta$
$-35\alpha'\beta'\gamma^3\delta$
$-35\alpha'\beta'\gamma\delta^3$
$+35\alpha'\beta'\gamma\delta''$
$+105\alpha'\beta'\gamma'\delta'$
$+105\alpha'\beta'\gamma''\delta$
$+70\alpha'\beta''\gamma\delta'$
$+140\alpha'\beta''\gamma'\delta$
$+70\alpha'\beta'''\gamma\delta$
$-21\alpha''\beta^3\gamma\delta$
$-21\alpha''\beta\gamma^3\delta$

$-21\alpha''\beta\gamma\delta^3$
$+21\alpha''\beta\gamma\delta''$
$+63\alpha''\beta\gamma'\delta'$
$+63\alpha''\beta\gamma''\delta$
$+84\alpha''\beta'\gamma\delta'$
$+168\alpha''\beta'\gamma'\delta$
$+126\alpha''\beta''\gamma\delta$
$+35\alpha'''\beta\gamma\delta'$
$+70\alpha'''\beta\gamma'\delta$
$+105\alpha'''\beta'\gamma\delta$
$+35^{iv}\beta\gamma\delta$

$+\alpha^3\beta\gamma\delta\epsilon^2$
$+\alpha\beta^3\gamma\delta\epsilon^2$
$+\alpha\beta\gamma^3\delta\epsilon^2$
$+2\alpha\beta\gamma\delta^3\epsilon^2$
$+\alpha\beta\gamma\delta\epsilon^4$
$-3\alpha\beta\gamma\delta\epsilon'^2$
$-4\alpha\beta\gamma\delta\epsilon\epsilon''$

$-12\alpha\beta\gamma\delta'\epsilon\epsilon'$
$-6\alpha\beta\gamma\delta''\epsilon^2$
$-15\alpha\beta\gamma'\delta\epsilon\epsilon'$
$-15\alpha\beta\gamma'\delta'\epsilon^2$
$-10\alpha\beta\gamma''\delta\epsilon^2$
$-18\alpha\beta'\gamma\delta\epsilon\epsilon'$
$-18\alpha\beta'\gamma\delta'\epsilon^2$
$-24\alpha\beta'\gamma'\delta\epsilon^2$
$-15\alpha\beta''\gamma\delta\epsilon^2$
$-21\alpha'\beta\gamma\delta\epsilon\epsilon'$
$-21\alpha'\beta\gamma\delta'\epsilon^2$
$-28\alpha'\beta\gamma'\delta\epsilon^2$
$-35\alpha'\beta'\gamma\delta\epsilon^2$
$-21\alpha''\beta\gamma\delta\epsilon^2$

$+\alpha\beta\gamma\delta\epsilon^2\theta^2$

$$\xi_6=\alpha\beta\gamma\delta\epsilon\,\frac{S^6}{6!}+$$

$S^7/7!$

$+\alpha\beta\gamma\delta\epsilon'$
$+2\alpha\beta\gamma\delta'\epsilon$
$+3\alpha\beta\gamma'\delta\epsilon$
$+4\alpha\beta'\gamma\delta\epsilon$
$+5\alpha'\beta\gamma\delta\epsilon$

$S^8/8!$

$-\alpha^3\beta\gamma\delta\epsilon$
$-\alpha\beta^3\gamma\delta\epsilon$
$-\alpha\beta\gamma^3\delta\epsilon$
$-\alpha\beta\gamma\delta^3\epsilon$
$-\alpha\beta\gamma\delta\epsilon^3$
$+\alpha\beta\gamma\delta\epsilon''$
$+3\alpha\beta\gamma\delta'\epsilon'$
$+3\alpha\beta\gamma\delta''\epsilon$
$+4\alpha\beta\gamma'\delta\epsilon'$
$+8\alpha\beta\gamma'\delta'\epsilon$
$+6\alpha\beta\gamma''\delta\epsilon$
$+5\alpha\beta'\gamma\delta\epsilon'$
$+10\alpha\beta'\gamma\delta'\epsilon$
$+15\alpha\beta'\gamma'\delta\epsilon$
$+10\alpha\beta''\gamma\delta\epsilon$
$+6\alpha'\beta\gamma\delta\epsilon'$
$+12\alpha'\beta\gamma\delta'\epsilon$
$+18\alpha'\beta\gamma'\delta\epsilon$
$+24\alpha'\beta'\gamma\delta\epsilon$
$+15\alpha''\beta\gamma\delta\epsilon$

$-\alpha\beta\gamma\delta\epsilon\theta^2$

$S^9/9!$

$-\alpha^3\beta\gamma\delta\epsilon'$
$-2\alpha^3\beta\gamma\delta'\epsilon$
$-3\alpha^3\beta\gamma'\delta\epsilon$
$-4\alpha^3\beta'\gamma\delta\epsilon$
$-18\alpha^2\alpha'\beta\gamma\delta\epsilon$
$-\alpha\beta^3\gamma\delta\epsilon'$
$-2\alpha\beta^3\gamma\delta'\epsilon$
$-3\alpha\beta^3\gamma'\delta\epsilon$

$-15\alpha\beta^2\beta'\gamma\delta\epsilon$
$-\alpha\beta\gamma^3\delta\epsilon'$
$-2\alpha\beta\gamma^3\delta'\epsilon$
$-12\alpha\beta\gamma^2\gamma'\delta\epsilon$
$-\alpha\beta\gamma\delta^3\epsilon'$
$-9\alpha\beta\gamma\delta^2\delta'\epsilon$
$-6\alpha\beta\gamma\delta\epsilon^2\epsilon'$
$+\alpha\beta\gamma\delta\epsilon'''$
$+4\alpha\beta\gamma\delta'\epsilon''$
$+6\alpha\beta\gamma\delta''\epsilon'$
$+4\alpha\beta\gamma\delta'''\epsilon$
$-4\alpha\beta\gamma\delta'\epsilon^3$
$-5\alpha\beta\gamma'\delta^3\epsilon$
$-5\alpha\beta\gamma'\delta\epsilon^3$
$+5\alpha\beta\gamma'\delta\epsilon''$
$+15\alpha\beta\gamma'\delta'\epsilon'$
$+15\alpha\beta\gamma'\delta''\epsilon$
$+10\alpha\beta\gamma''\delta\epsilon'$
$+20\alpha\beta\gamma''\delta'\epsilon$
$+10\alpha\beta\gamma'''\delta\epsilon$

$-6\alpha\beta'\gamma^3\delta\epsilon$
$-6\alpha\beta'\gamma\delta^3\epsilon$
$-6\alpha\beta'\gamma\delta\epsilon^3$
$+6\alpha\beta'\gamma\delta\epsilon''$
$+18\alpha\beta'\gamma\delta'\epsilon'$
$+48\alpha\beta'\gamma'\delta'\epsilon$
$+24\alpha\beta'\gamma'\delta\epsilon'$
$+18\alpha\beta'\gamma\delta''\epsilon$
$+36\alpha\beta'\gamma''\delta\epsilon$
$+15\alpha\beta''\gamma\delta\epsilon'$
$+30\alpha\beta''\gamma\delta'\epsilon$
$+45\alpha\beta''\gamma'\delta\epsilon$
$+20\alpha\beta'''\gamma\delta\epsilon$
$-7\alpha'\beta^3\gamma\delta\epsilon$
$-7\alpha'\beta\gamma^3\delta\epsilon$
$-7\alpha'\beta\gamma\delta^3\epsilon$
$-7\alpha'\beta\gamma\delta\epsilon^3$
$+7\alpha'\beta\gamma\delta\epsilon''$
$+21\alpha'\beta\gamma\delta'\epsilon'$
$+28\alpha'\beta\gamma'\delta\epsilon'$

$+56\alpha'\beta\gamma'\delta'\epsilon$
$+21\alpha'\beta\gamma\delta''\epsilon$
$+42\alpha'\beta\gamma''\delta\epsilon$
$+36\alpha'\beta'\gamma\delta\epsilon'$
$+70\alpha'\beta'\gamma\delta'\epsilon$
$+105\alpha'\beta'\gamma'\delta\epsilon$
$+70\alpha'\beta''\gamma\delta\epsilon$
$+21\alpha''\beta\gamma\delta\epsilon'$
$+42\alpha''\beta\gamma\delta'\epsilon$
$+63\alpha''\beta\gamma'\delta\epsilon$
$+84\alpha''\beta'\gamma\delta\epsilon$
$+35\alpha'''\beta\gamma\delta\epsilon$

$-3\alpha\beta\gamma\delta\epsilon'\theta^2$
$-3\alpha\beta\gamma\delta\epsilon\theta\theta'$
$-4\alpha\beta\gamma\delta'\epsilon\theta^2$
$-5\alpha\beta\gamma'\delta\epsilon\theta^2$
$-6\alpha\beta'\gamma\delta\epsilon\theta^2$
$-7\alpha'\beta\gamma\delta\epsilon\theta^2$

$$\xi_7 = \alpha\beta\gamma\delta\epsilon\theta . \frac{S^7}{7!} +$$

$S^8/8!$	$S^9/9!$	$+\alpha\beta\gamma\delta\epsilon\theta''$	$+15\alpha\beta\gamma'\delta'\epsilon\theta$	$+14\alpha'\beta\gamma\delta\epsilon'\theta$
		$+3\alpha\beta\gamma\delta\epsilon'\theta'$	$+10\alpha\beta\gamma''\delta\epsilon\theta$	$+21\alpha'\beta\gamma\delta'\epsilon\theta$
$+\alpha\beta\gamma\delta\epsilon\theta'$	$-\alpha^3\beta\gamma\delta\epsilon\theta$	$+3\alpha\beta\gamma\delta\epsilon''\theta$	$+6\alpha\beta'\gamma\delta\epsilon\theta'$	$+28\alpha'\beta\gamma'\delta\epsilon\theta$
$+2\alpha\beta\gamma\delta\epsilon'\theta$	$-\alpha\beta^3\gamma\delta\epsilon\theta$	$+4\alpha\beta\gamma\delta'\epsilon\theta'$	$+12\alpha\beta'\gamma\delta\epsilon'\theta$	$+35\alpha'\beta'\gamma\delta\epsilon\theta$
$+3\alpha\beta\gamma\delta'\epsilon\theta$	$-\alpha\beta\gamma^3\delta\epsilon\theta$	$+8\alpha\beta\gamma\delta'\epsilon'\theta$	$+18\alpha\beta'\gamma\delta'\epsilon\theta$	$+21\alpha''\beta\gamma\delta\epsilon\theta$
$+4\alpha\beta\gamma'\delta\epsilon\theta$	$-\alpha\beta\gamma\delta^3\epsilon\theta$	$+6\alpha\beta\gamma\delta''\epsilon\theta$	$+24\alpha\beta'\gamma'\delta\epsilon\theta$	
$+5\alpha\beta'\gamma\delta\epsilon\theta$	$-\alpha\beta\gamma\delta\epsilon^3\theta$	$+5\alpha\beta\gamma'\delta\epsilon\theta'$	$+15\alpha\beta''\gamma\delta\epsilon\theta'$	$-\alpha\beta\gamma\delta\epsilon\theta\phi^2$
$+6\alpha'\beta\gamma\delta\epsilon\theta$	$-\alpha\beta\gamma\delta\epsilon\theta^3$	$+10\alpha\beta\gamma'\delta\epsilon'\theta$	$+7\alpha'\beta\gamma\delta\epsilon\theta'$	

$$\xi_8 = \alpha\beta\gamma\delta\epsilon\theta\phi . \frac{S^8}{8!} + \left(\alpha\beta\gamma\delta\epsilon\theta\phi' + 2\alpha\beta\gamma\delta\epsilon\theta'\phi + 3\alpha\beta\gamma\delta\epsilon'\theta\phi + 4\alpha\beta\gamma\delta'\epsilon\theta\phi\right.$$

$$\left. + 5\alpha\beta\gamma'\delta\epsilon\theta\phi + 6\alpha\beta'\gamma\delta\epsilon\theta\phi + 7\alpha'\beta\gamma\delta\epsilon\theta\phi\right)\frac{S^9}{9!} +$$

$$\xi_9 = \alpha\beta\gamma\delta\epsilon\theta\phi\psi . \frac{S^9}{9!} +$$

PARAMETRIC CO-EFFICIENTS IN THE DIFFERENTIAL GEOMETRY OF CURVES

BY

SYAMADAS MUKHOPADHYAYA.

V. PRINCIPAL DIRECTIONS AND CURVATURES AT A SINGULAR POINT OF A CURVE IN AN N-SPACE.*

1. Nature and Conditions of Singularity.

If $x_1, x_2, .., x_n$ be the n co-ordinates of a point P in an n-space, referred to any system of orthogonal axes and if $x_1, x_2, .., x_n$ be functions of a single parameter t, then the locus of P is a curve in an n-space. We shall suppose that at a given point P, the derivates of $x_1, x_2, .., x_n$ of any required order, with respect to t, exist.

In general $r+1$ consecutive points on the curve at P will determine an osculating r-space. The osculating r-space at P may happen to be stationary, that is, pass through $r+2$ consecutive points. The point P is in that case a singular point. The r^{th} curvature will vanish at P and the usual expressions for the higher principal curvatures as also those for the direction cosines of the higher principal axes at P will take indeterminate forms. It is the object of this paper to obtain determinate expressions for these.

Suppose the osculating r-space at P is stationary for a given value of r. To illustrate our methods, which are quite general, we will choose first some simple value of r, say r equal to 2. That is, we will suppose the osculating plane at P to be stationary and consequently the second curvature to vanish.

The conditions that the osculating plane at P may be stationary are obviously,

$$\left\| \begin{matrix} x_1', & x_2', & .. & x_n' \\ x_1'', & x_2'', & .. & x_n'' \\ x_1''', & x_2''', & .. & x_n''' \end{matrix} \right\| = 0 \qquad (i)$$

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* From Bulletin, Calcutta Mathematical Society, Vol. V, 1913.

Hence the matrix product,

$$\begin{vmatrix} x_1', & x_2', & \ldots & x_n' \\ x_1'', & x_2'', & \ldots & x_n'' \\ x_1''', & x_2''', & \ldots & x_n''' \end{vmatrix} \begin{vmatrix} x_1^{(p)}, & x_2^{(p)}, & \ldots, & x_n^{(p)} \\ x_1^{(q)}, & x_2^{(q)}, & \ldots, & x_n^{(q)} \\ x_1^{(r)}, & x_2^{(r)}, & \ldots, & x_n^{(r)} \end{vmatrix}$$

vanishes for all values of p, q, r.

For this product we have adopted the notation

$$[1,2,3 \mid p,q,r],$$

so that we have

$$[1,2,3, \mid p,q,r]=0.$$

Similarly, as consequence of equations (i),

$$\begin{vmatrix} x_1', & x_2', & \ldots, & x_n' \\ x_1'', & x_2'', & \ldots, & x_n'' \\ x_1''', & x_2''', & \ldots, & x_n''' \\ x_1^{(l)}, & x_2^{(l)}, & \ldots, & x_n^{(l)} \end{vmatrix} \begin{vmatrix} x_1^{(p)}, & x_2^{(p)}, & \ldots, & x_n^{(p)} \\ x_1^{(q)}, & x_2^{(q)}, & \ldots, & x_n^{(q)} \\ x_1^{(r)}, & x_2^{(r)}, & \ldots, & x_n^{(r)} \\ x_1^{(s)}, & x_2^{(s)}, & \ldots, & x_n^{(s)} \end{vmatrix}$$

or, in our notation,

$$[1,\ 2,\ 3,\ l \mid p,\ q,\ r,\ s]=0,$$

where p, q, r, s are any positive integers. In fact, parametric co-efficients of the above type involving 1, 2, 3 in the same section, vanish.

Further, if F be any function of the co-ordinates of P, the matrix product

$$\begin{vmatrix} x_1', & x_2', & \ldots & x_n', & 0 \\ x_1'', & x_2'', & \ldots, & x_n'', & 0 \\ x_1''', & x_2''', & \ldots, & x_n''', & 0 \\ 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} \begin{vmatrix} x_1^{(p)}, & x_2^{(r)}, & \ldots & x_n^{(p)}, & F^{(p)} \\ x_1^{(q)}, & x_2^{(q)}, & \ldots, & x_n^{(q)}, & F^{(q)} \\ x_1^{(r)}, & x_2^{(r)}, & \ldots & x_n^{(r)}, & F^{(r)} \\ x_1^{(s)}, & x_2^{(s)}, & \ldots & x_n^{(s)}, & F^{(s)} \end{vmatrix}$$

for which we have adopted the notation

$$[p,\ q,\ r,s \mid 1,\ 2,\ 3,\ F]$$

will vanish. In fact, all *mixed parametric co-efficients* of the above type, involving 1, 2, 3 *in the same section with F*, will vanish.

Again if F be equal to one of the quantities $x_1, x_2, .., x_n$, the matrix product

$$\begin{vmatrix} x_1^{(p)}, & x_2^{(p)}, & .., & x_n^{(p)}, & 0 \\ x_1^{(q)}, & x_2^{(q)}, & .., & x_n^{(q)}, & 0 \\ x_1^{(r)}, & x_2^{(r)}, & .., & x_n^{(r)}, & 0 \\ 0, & 0, & .., & 0, & 1 \end{vmatrix} \begin{vmatrix} x'_1, & x'_2, & .., & x'_n, & F' \\ x''_1, & x''_2, & .., & x''_n, & F'' \\ x'''_1, & x'''_2, & .., & x'''_n, & F''' \\ x_1^{(l)}, & x_2^{(l)}, & .., & x_n^{(l)}, & F^{(l)} \end{vmatrix}$$

which in our notation is

$$[1, 2, 3, l \mid p, q, r, F]$$

will vanish, obviously. In fact, all *mixed* parametric co-efficients of the above type, involving 1, 2, 3 in the section which *does not* contain F, will vanish, *provided* F stands for one of the quantities $x_1, x_2, .., x_n$.

Also, if we remember that $[1, 2, 3]$, $[1, 2, 3, l]$, etc. stand for square root of

$$[1, 2, 3 \mid 1, 2, 3], [1, 2, 3, l \mid 1, 2, 3, l], \&c.,$$

we see that

$$[1, 2, 3] = 0 \quad [1, 2, 3, l] = 0, \&c.$$

Hence we conclude that if the osculating plane at P be *stationary*, *all parametric co-efficients, pure and mixed, involving the indices* 1, 2, 3 *in one section*, will vanish, provided F, when it occurs, stands for one of the quantities $x_1, x_2, .., x_n$, as will be the case in the following investigations.

2. Formulation of the Problem.

The direction cosines of the principal axes at any point P whose co-ordinates are $x_1, x_2, .., x_n$, are

$$\triangle_1(x_1), \triangle_1(x_2), .., \triangle_1(x_n)$$

$$\triangle_2(x_1), \triangle_2(x_2), .., \triangle_2(x_n)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\triangle_n(x_1), \triangle_n(x_2), .., \triangle_n(x_n),$$

where $\triangle_1(x_1)=\frac{x_1'}{[1]}$, [1] standing for the positive square root of

$x_1'^2+x_2'^2+.+x_n'^2$,

$$\left.\begin{aligned}
&\triangle_2(x_1)=\frac{[1,2\mid 1,x_1]}{[1,2][1]}\\
&\triangle_3(x_1)=\frac{[1,2,3\mid 1,2,x_1]}{[1,2,3][1,2]}\\
&\dots\quad\dots\quad\dots\quad\dots\\
&\triangle_n(x_1)=\frac{[1,2,..n\mid 1,2,..n-1,x_1]}{[1,2,..n][1,2,..n-1]}\ ,\quad \&c.^{*}
\end{aligned}\right\}\ (ii)$$

If the osculating plane at P be stationary we have

$[1,2,3]=0,\ [1,2,3,4]=0,\ [1,2,3\mid 1,2,x_1]=0,$

$[1,2,3,4\mid 1,2,3,x_1]=0,$ &c.

Hence the direction cosines of the third and higher principal axes take indeterminate forms. We shall obtain determinate expressions for these direction cosines, namely,

$$\left.\begin{aligned}
&\triangle_3(x_1)=\frac{[1,2,4\mid 1,2,x_1]}{[1,2,4][1,2]}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\triangle_n(x_1)=\frac{[1,2,4,..n+1\mid 1,2,4,..n,x_1]}{[1,2,4,..n+1][1,2,4,..n]}
\end{aligned}\right\}\ (iii)$$

which are obtained from (*ii*) by increasing all the indices, from 3 upwards, by unity.

Again the expressions for the principal curvatures $u_1,\ u_2,\ u_3,\ ..\ u_{n-1}$, at P, are†

$$u_1=\frac{[1,2]}{[1]^3},\ u_2=\frac{[1,2,3]}{[1,2]^2},\ u_3=\frac{[1,2,3,4][1,2]}{[1,2,3]^2[1]},\dots$$

$$u_{n-1}=\frac{[1,2,..n][1,2,..n-2]}{[1,2,..n-1]^2[1]}\qquad (iv)$$

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[*] *Vide* Bulletin C. M. S., *Vol. I, pages* 199 and 238.

[†] *Vide* Bulletin C. M. S., *Vol. I, page* 200.

If the osculating plane at P be stationary, $u_2=0$, and $u_3, u_4, .. u_{n-1}$ take indeterminate forms. We shall obtain determinate expressions for $u_3, u_4, .. u_{n-1}$, namely,

$$u_3=\frac{1}{2!}\frac{[1,2,4,5][1,2]}{[1,2,4]^2[1]}.$$

$$u_4=\frac{(2!)^2}{3!1!}\frac{[1,2,4,5,6][1,2,4]}{[1,2,4,5]^2[1]}$$

……

$$u_{n-1}=\frac{(n-3)!^2}{(n-2)!(n-4)!}\cdot\frac{[1,2,4,5,..,n+1][1,2,4,5,..,n-1]}{[1,2,4,5,..,n]^2[1]} \qquad (v)$$

which are obtained from (*iv*), by increasing all the numerical indices, from 3 upwards, by unity, and introducing a suitable numerical co-efficient.

3. Special Methods of Differentiation.

We have already explained the principles on which parametric co-efficients are differentiated. The simple rule* is

Increase all the numerical indices, successively, by unity, and take the sum of the expressions thus obtained, the functionals, if any, being left untouched.

Let D, placed before a parametric co-efficient, indicate differentiation with respect to the parameter t, and D: placed inside the brackets before some of the numerical indices in either of the two sections of a parametric co-efficient indicate successive increase by unity of the numerical indices which follow it in the section.

Thus, $\quad D[1, 2, 3, 4 \mid 1, 2, 3, F]$

$$=[D: 1, 2, 3, 4 \mid 1, 2, 3, F]+[1, 2, 3, 4 \mid D: 1, 2, 3, F]$$

$$=[1, 2, 3, 5 \mid 1, 2, 3, F]+[1, 2, 3, 4 \mid 1, 2, 4, F],$$

as the parametric co-efficient vanishes if two of the indices in the same section are alike.

It may be observed that a formula, analogous to Leibnitz's formula for r^{th} differential co-efficient of the product of two functions.

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* *Vide* Bulletin C. M. S., *Vol. I*, *page* 237.

applies to the r^{th} differential co-efficient of a parametric co-efficient, pure or mixed, looked upon as the product of its two sections.

Thus, $D^r[1, 2, 3 \mid 1, 2, F]$

$=[D^r : 1, 2, 3 \mid 1, 2, F]+r\,[D^{r-1} : 1, 2, 3 \mid D : 1, 2, F]+\&c.$

$D^r\,[1, 2, 3]^2=D^r\,[1, 2, 3 \mid 1, 2, 3]$

$=[D^r : 1, 2, 3 \mid 1, 2, 3]+r\,[D^{r-1} : 1, 2, 3 \mid D : 1, 2, 3]+\&c.$

Again, we may write

$D:1, 2, 3, 4, 5=1, 2, 3, 4, 6=0$, if $[1, 2, 3]=0$, meaning thereby that a parametric co-efficient having as one of its sections $D : 1, 2, 3, 4, 5$ will vanish, if $[1, 2, 3]=0$.

In a similar sense, if $[1, 2, 3]=0$,

$D^2 : 1, 2, 3, 4, 5=D : 1, 2, 3, 4, 6=1, 2, 3, D : 4, 6=0.$

$D^3 : 1, 2, 3, 4, 5=D : 1, 2, 3, D : 4, 6$

$=1, 2, 3, D^2 : 4, 6+1, 2, 4, D : 4, 6$

$=1, 2, 4, 5, 6.$

In general $D^r : 1, 2, 3, .., p=0$, if r be less than $p-2$ and

$D^r : 1, 2, 3, .., p=1, 2, 4, .., p+1$, if $r=p-2$, provided $[1, 2, 3]=0$.

We will apply the above principles to calculate

$D^r[I, 2, 3, .., p \mid 1, 2, 3, .., p-1, F]$

when $F=x_1, x_2, ..$ or x_n and $[1, 2, 3]=0$.

We have $D^r\,[1, 2, 3, .., p \mid 1, 2, 3, .., p-1, F]$

$=[1, 2, 3, .., p \mid D^r : 1, 2, 3, .., p-1, F]+r\,[D : 1, 2, 3, .., p \mid D^{r-1} : 1, 2, 3, .., p-1, F]+\frac{r(r-1)}{1\cdot 2}\,[D^2 : 1, 2, 3, .., p \mid D^{r-2} : 1, 2, 3, .., p-1, F]+\&c., =0$, if r be less than $2p-5$.

For, $D^r : 1, 2, 3, .., p$ vanishes if r be less than $p-2$, and $D^r : 1, 2, 3, .., p-1, F$ vanishes if r be less than $p-3$, and because the sum of the indices of $D :$ in the two sections of each term is equal to r, one of the two sections will vanish in each term, if r be less than $2p-5$.

If however, r be equal to $2p-5$, there will be one term, namely,

$$\frac{(2p-5)!}{(p-2)!\ (p-3)!}[D^{p-2}: 1, 2, 3, .., p \mid D^{p-3}: 1, 2, 3, .. p-1, F]$$

which will not vanish.

Hence $D^r[1, 2, 3, .., p \mid 1, 2, 3, p-1, F]=0$ or $\dfrac{(2p-5)}{(p-2)!\,(p-3)!}$ $[1, 2, 4, .. p+1 \mid 1, 2, 4, .., p, F]$, according as r is less or equal to $2p-5$. (*vi*)

Similarly,

$$D^r\ [1, 2, 3, .., p]^2=0 \text{ or } \frac{(2p-4)!}{(p-2)!\,(p-2)!}\ [1, 2, 4, .., p+1]^2 \qquad (vii)$$

according as r is less than or equal to $2p-4$.

Again, $D^r[1, 2, 3, ., p \mid 1, 2, 3, .. p-1, F]^2$

$=[1, 2, 3, .. p \mid 1, 2, 3, .. p-1, F]\,D^r[1, 2, 3, .. p \mid 1, 2, 3, ..p-1, F]+rD[1, 2, 3, .. p \mid 1, 2, 3, .. p-1, p]D^{r-1}[1, 2, 3, .. p \mid 1, 2, 3, .. p-1, F] + \dfrac{p(p-1)}{1.\ 2}\ D^2[1, 2, 3, .. p \mid 1, 2, 3, .. p-1, F]D^{r-2}[1, 2, 3, .. p \mid 1, 2, 3, p-1, F] +$ &c.,

$$=0 \text{ or } \frac{(4p-10)!}{\{(p-2)!\,(p-3)!\}^2}\ [1, 2, 4, .. p+1 \mid 1, 2, 4, .. p, P]^2 \quad (viii)$$

according as r is less than or equal to $4p-10$.

Similarly, $D^r\ [1, 2, 3, .. p]^2[1, 2, 3, .. q]^2$

$$=0 \text{ or } \frac{(2p+2q-8)!}{\{(p-2)!\,(q-2)!\}^2}[1, 2, 4, .. p+1]^2[1, 2, 4, .. q+1]^2$$

according as r is less than or equal to $2p+2q-8$ (*ix*)

As a particular case of the above, we have

$$D^r[1, 2, 3, ., p]^4=0 \text{ or } \frac{(4p-8)!}{\{(p-2)!\}^4}[1, 2, 4, .., p+1]^4$$

according as r is less than or equal to $4p-8$.

4. Demonstrations.

Now we can evaluate the indeterminate expressions in *(ii)* and *(iv)*, when $[1, 2, 3] = 0$.

To evaluate $\Delta_3(x) = \dfrac{[1, 2, 3 \mid 1, 2, x_1]}{[1, 2, 3][1, 2]}$ we have

$$\frac{[1, 2, 3 \mid 1, 2, x_1]^2}{[1, 2, 3]^2} = \frac{D^2[1, 2, 3 \mid 1, 2, x_1]^2}{D^2[1, 2, 3]^2}$$

$$= \frac{2![1, 2, 4 \mid 1, 2, x_1]^2}{2![1, 2, 4]^2} = \frac{[1, 2, 4 \mid 1, 2, x_1]^2}{[1, 2, 4]^2} \text{ by } (vii) \text{ and } (viii).$$

Therefore, $\Delta_3\ (x_1) = \dfrac{[1, 2, 4 \mid 1, 2\ x_1]}{[1, 2, 4][1, 2]}$.

Similarly, to evaluate

$$\Delta_4(x_1) = \frac{[1, 2, 3, 4 \mid 1, 2, 3, x_1]}{[1, 2, 3, 4][1, 2, 3]}, \text{ we have}$$

$$\frac{[1, 2, 3, 4 \mid 1, 2, 3 x_1]^2}{[1, 2, 3, 4]^2[1, 2, 3]^2} = \frac{D^6[1, 2, 3, 4 \mid 1, 2, 3\ x_1]^2}{D^6[1, 2, 3, 4]^2[1, 2, 3]^2}$$

$$= \frac{\dfrac{6!}{(2!)^2}[1, 2, 4, 5 \mid 1, 2,\ 4\ x_1]^2}{\dfrac{6!}{(2!)^2}[1, 2, 4, 5]^2[1, 2, 4]^2} = \frac{[1, 2, 4, 5 \mid 1, 2, 4, x_1]^2}{[1, 2, 4, 5]^2[1, 2, 4]^2}$$

by *(viii)* and *(ix)*.

Therefore, $\Delta_4\ (x_1) = \dfrac{[1, 2, 4, 5 \mid 1, 2, 4\ x_1]}{[1, 2, 4, 5][1, 2, 4]}$, &c., &c.

Next, to evaluate $u_3 = \dfrac{[1, 2, 3, 4]\ [1, 2]}{[1, 2, 3]^2\ [1]}$, we have

$$\frac{[1, 2, 3, 4]^2}{[1, 2, 3]^4} = \frac{D^4[1, 2, 3,\ 4]^2}{D^4[1, 2, 3]^4} = \frac{\dfrac{4!}{2!2!}[1, 2,\ 4,\ 5]^2}{4![1, 2, 4]^4}$$

$$= \frac{[1, 2, 4, 5]^2}{2!2![1, 2, 4]^2} \text{ by } (vii) \text{ and } (x).$$

Therefore, $u_3 = \dfrac{[1, 2, 4, 5]\ [1, 2]}{2![1, 2, 4]^2[1]}$.

Similarly, to evaluate $u_{n-1}=\dfrac{[1,2,..n][1,2,..,n-2]}{[1,2,..n-1]^2[1]}$

we have $\dfrac{[1,2,..n]^2[1,2,..,n-2]^2}{[1,2,..,n-1]^2}$

$$\frac{D^{4n-12}[1,2,..n]^2[1,2,..,n-2]^2}{D^{4n-12}[1,2,..n-1]^2}$$

$$=\frac{\dfrac{(4n-12)!}{\{(n-2)!\,(n-4)!\}^2}[1,2,4,..n+1]^2[1,2,4,..n-1]^2}{\dfrac{(4n-12)!}{(n-3)!^4}[1,2,4,..,n]^4}$$

$$=\frac{(n-3)!^4}{\{(n-2)!\,(n-4)!\}^2}\cdot\frac{[1,2,4,..n+1]^2[1,2,4,..n-1]^2}{[1,2,4,..,n]^4}$$

Therefore

$$u_{n-1}=\frac{(n-3)!^2}{(n-2)!\;(n-4)!}\,\frac{[1,2,4,..,n+1][1,2,4,..n-1]}{[1,2,4,..n]^2[1]}$$

5. General Case when $r=m-1$.

Next suppose that $r=m-1$, so that the osculating $(m-1)$-space is stationary at P; consequently the $m-1$th curvature vanishes and $[1,2,..m]=0$. The direction cosines of the mth and higher principal axes become indeterminate in form as also the expressions for the mth and higher principal curvatures. We can however by adopting the method of the foregoing articles obtain determinate expressions for the direction cosines of the mth and higher principal axes as also for mth and higher principal curvatures.

The expressions for the direction cosines of the mth and higher principal axes are found to be

$$\triangle_m(x_1)=\frac{[1,2,..,m-1,m+1\,|\,1,2,..,m-1,x_1]}{[1,2,..m-1,m+1][1,2,..,m-1]}$$

$$\triangle_{m+1}(x_1)=\frac{[1,2,..m-1,m+1,m+2\,|\,1,2,..,m-1,m+1,x_1]}{[1,2,..m-1,m+1,m+2][1,2,..m-1,m+1]}$$

... ...

$$\triangle_n(x_1)=\frac{[1,2,..m-1,m+1,..,n+1\,|\,1,2,..,m-1,m+1,..,n,x_1]}{[1,2,..m-1,m+1,..,n+1][1,2,..m-1,m+1,..,n]}$$

etc.

The expressions for the *m*th and higher principal curvatures are

$$u_m=\frac{1}{2!}\,\frac{[1,2,..\,m-1,m+1,m+2][1,2,..\,m-1]}{[1,2,..\,m-1,m+1]^2[1]}$$

$$u_{m+1}=\frac{(2!)^2}{3!1!}\,\frac{[1,2,..\,m-1,m+1,..\,m+3][1,2,..\,m-1,m+1]}{[1,2,..\,m-1,m+1,m+2]^2\,[1]}$$

... ...

$$u_{n-1}=\frac{(n-m)!^2}{(n-m+1)!\,(n-m-1)!}$$

$$\frac{[1,2,..\,m-1,m+1,..\,n+1][1,2,..\,m-1,m+1,..\,n-1]}{[1,2,..\,m-1,m+1,..\,n]^2\,[1]}$$

The methods of this paper can be easily extended to the case of multiple singularities.

PARAMETRIC CO-EFFICIENTS IN THE DIFFERENTIAL GEOMETRY OF CURVES

BY

SYAMADAS MUKHOPADHYAYA.

VI. ON NUL PARAMETRIC CO-EFFICIENTS AND OSCULATING SPHERICS TO A CURVE IN AN N-SPACE.*

1. Let $x_r (r=1, 2, .., n)$ be the n co-ordinates of a point, referred to n mutually orthogonal axes in an n-space, which may be called the point x. If x depend on a single variable parameter t the locus of x will be in general a curve in the n-space.

Suppose $x_r^{(p)}$ denote the p^{th} derivative of x_r with respect to t. Then

$$\Sigma x_r^{(p)} x_r^{(q)}, \ (r=1, 2, .., n)$$

has been called a parametric co-efficient of class I. It is symbolically indicated by (p, q), where p and q are positive integers. We will now introduce and interpret parametric co-efficients in which one or both of the indices p and q are zero. Such parametric co-efficients may be called *nul parametric co-efficients.*

The parametric co-efficient (p, q) is evidently the limit of

$$\Sigma (X_r - x_r)^{(p)} (X_r - x_r)^{(q)}, \ (r=1, 2, .., n)$$

when $X=x$, where X is a *variable* point on the curve in the neighbourhood of the *fixed* point x. We will define the nul parametric co-efficients (o, p) and (o, o) as the limits of

$$\Sigma (X_r - x_r) (X_r - x_r)^{(p)} \text{and } \Sigma (X_r - x_r) (X_r - x_r)$$

respectively, when $X=x$.

As (o, p) and (o, o) are both constantly zero along the curve, their derivatives of any order with respect to t are also zero. We may however derive from each of them a set of functions by making the rules for differentiation of a parametric co-efficient when both indices are positive integers, also applicable to them.

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* From Bulletin, Calcutta Mathematical Society, Vol. VII, 1915.

The rules for differentiating (p, q) give, as we know,

$$(p, q)' = (p, q+1) + (p+1, q)$$

$$(p, q)^{(r)} = (p, q+r) + r\,(p+1, q+r-1)$$
$$+ \frac{r(r-1)}{1 \cdot 2}(p+2, q+r-2) + \ldots + (p+r, q)$$

We will define

$$(o, p)_1 = (o, p+1) + (1, p) = (1, p)$$

$$(o, p)_r = (o, p+r) + r(1, p+r-1) + \frac{r(r-1)}{1 \cdot 2}(2, p+r-2)$$
$$+ \ldots + (r, p)$$

$$= r(1, p+r-1) + \frac{r(r-1)}{1 \cdot 2}(2, p+r-2) + \ldots + (r, \ \ )$$

and $(o, o)_1 = (0, 1) + (1, 0) = 0$

$$(o, o)_2 = (0, 2) + 2(1, 1) + (2, 0) = 2(1, 1)$$

$$(o, o)_3 = (0, 3) + 3(1, 2) + 3(2, 1) + (3, 0) = 6(1, 2)$$

……………………………………………………………

$$(o, o)_r = (0, r) + r(1, r-1) + \frac{r(r-1)}{1,2}(2, r-2) + \ldots + (r, 0)$$

$$= 2(1, r-1) + \frac{2\,r(r-1)}{1,\,2}(2, r-2) + \ldots \qquad (i)$$

It is evident from the definitions of $(o, p)_r$ and $(o, o)_r$, that

$$(o, p)_r = \mathrm{Lt}_{X=x}\left(\frac{d}{dt}\right)^r \Sigma\left(X_r - x_s\right)\left(X_s - x_s\right)^{(p)}, \ (s = 1, 2, .., n)$$

$$\text{and } (o, o)_r = \mathrm{Lt}_{X=x}\left(\frac{d}{dt}\right)^r \Sigma\left(X_s - x_s\right)^2, \ (s = 1, 2, .., n)$$

where X is a *variable* point in the neighbourhood of the *fixed* point x.

In Paper III we have defined $\Delta_r\,(p)$ and in Paper IV we have given a table of values of $\Delta_r\,(p)$ as also the method of calculating the table. In Paper III, equation (C_1), we have

$$(p, q) = \Delta_1(p)\,\Delta_1\,(q) + \Delta_2(p)\,\Delta_2(q) + \ldots + \Delta_p\,(p)\,\Delta_p\,(q)$$

where p is a positive integer not less than q. Thus from the table of values of $\Delta_r\,(p)$ and equation (C_1) we can calculate $(o, p)_r$ and $(o, o)_r$.

2. We will use the abbreviations w and w_r for $\frac{1}{2}(o, o)$ and $\frac{1}{2}(o, o)_r$ and introduce an interesting co-efficient $\triangle_r(w)$, which is defined equal to

$$\begin{vmatrix} (1, 1), (1, 2), \ldots, (1, r-1), w_1 \\ (2, 1), (2, 2), \ldots, (2, r-1), w_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ (r, 1), (r, 2), \ldots, (r, r-1), w_r \end{vmatrix} \div [1, 2, \ldots, r][1, 2, \ldots, r-1] \qquad (ii)$$

which may be more compactly written as

$$\frac{[1, 2, \ldots, r \mid 1, 2, \ldots, r-1, w]}{[1, 2, \ldots, r][1, 2, \ldots, {}_r-1]}.$$

Here it may be noticed that $w_1, w_2, \ldots, w_r$ are not successive derivatives of w with respect to t but are derived from w in the manner already explained. In fact, we have by definition

$$w_r = r(1, r-1) + \frac{r(r-1)}{1.2}(2, r-2) + \ldots + r(r-1, 1)$$

$$w_{r+1} = (r+1)(1, r) + \frac{(r+1)r}{1.2}(1, r-1) + \ldots + (r+1)(r, 1).$$

Therefore

$$w'_r = r(1, r) + \frac{r(r-1)}{1.2}(2, r-1) + \ldots + r(r-1, 2)$$

$$+ r(2, r-1) + \ldots + \frac{r(r-1)}{1.2}(r-1, 2) + r(r, 1)$$

$$= r(1, r) + \frac{(r+1)r}{1.2}(2, r-1) + \ldots + \frac{(r+1)r}{1 \cdot 2}(r-2, 2) + r(r, 1)$$

$$= w_{r+1} - (1, r) - (r, 1) = w_{r+1} - 2(1, r) \qquad (iii)$$

Thus the first derived of w_r with respect to t is not w_{r+1} but $w_{r+1} - 2(1, r)$.

If $w_2, w_3 \ldots w_{r+1}$ were the first derived of $w_1, \ldots, w_r$, respectively, with respect to s then we could apply formula E of paper III, namely,

$$\triangle'_r(w) = \triangle_{r+1}(w).\, u_r - \triangle_{r-1}(w).\, u_{r-1}$$

$$\text{or } \triangle_{r+1}(w) = \frac{\triangle'_r(w) + \triangle_{r-1}(w).\, u_{r-1}}{u_r} \qquad \text{(E)}$$

by which we could successively and rapidly calculate $\triangle_3(w)$, $\triangle_4(w)$, $\triangle_5(w)$,...., knowing $\triangle_1(w)$ and $\triangle_2(w)$.

Fortunately, however, we can shew that Formula E holds even when $w_2, w_3, \ldots, w_r$, are not the successive derivatives of $w_1, w_2, \ldots, w_r$, respectively, but are defined in the way we have already done.

We have

$$\triangle_r (w) = c_1 w_1 + c_2 w_2 + \ldots + c_r w_r, \text{ by } (ii)$$

where $c_1, c_2, \ldots, c_r$ do not involve w.

Therefore

$$\begin{aligned}\triangle'_r (w) &= c'_1 w_1 + c'_2 w_2 + \ldots + c'_r w_r \\ &\quad + c_1 w' + c_2 w'_2 + \ldots\ldots c_r w'_r \\ &= c'_1 w_1 + c'_2 w_2 + \ldots + c'_r w_r \\ &\quad + c_1 w_2 + c_2 w_3 + \ldots + c_r w_{r+1} \\ &\quad - 2 c_1 (1, 1) - 2 c_2 (1, 2) - \ldots - 2 c_r (1, r) \text{ by } (iii) \\ &= c'_1 w'_1 + c'_2 w_2 + \ldots + c'_r w_r \\ &\quad + c_1 w_2 + c_2 w_3 + \ldots + c_r w_{r+1}\end{aligned}$$

For, $c_1 (1, 1) + c_2 (1, 2) + \ldots + c_r (1, r)$

$$= \begin{vmatrix} (1, 1), & (1, 2), & \ldots, & (1, r-1), & (1, 1) \\ (2, 1), & (2, 2), & \ldots & (2, r-1), & (2, 1) \\ \cdot & \cdot & & \cdot & \cdot \\ (r, 1), & (r, 2), & \ldots, & (r, r-1), & (r, 1) \end{vmatrix} \div [1, 2, \ldots, r][1, 2, \ldots, r-1]$$

$$= 0 \qquad \text{by } (ii)$$

Thus $\triangle'_r (w)$ would be obtained from $\triangle_r (w)$, if we simply wrote $w_2, w_3, \ldots, w_{r+1}$ in the place of the first derived of $w_1, w_2, \ldots, w_r$, with respect to s, in the process of differentiation. Formula E therefore holds for $\triangle_r (w)$ and enables us to calculate $\triangle_3(w)$, $\triangle_4(w)$, $\triangle_5(w)$,...., from knowledge of $\triangle_1(w)$ and $\triangle_2(w)$.

It may be observed that by Formula B_1 of Paper III we have

$$w_r = \triangle_1(r)\triangle_1(w) + \triangle_2(r)\triangle_2(w) + \ldots + \triangle_r(r)\triangle_r(w).$$

Thus w_r may be also calculated from tables of values of $\triangle_r(p)$ and $\triangle_r(w)$. A table of values of $\triangle_r(p)$ has been already given with

Paper IV. The method of working out a table of values for $\triangle_r(w)$ we proceed to consider.

We will take the variable parameter t to be the arc length s, so that $[1]=1$ where $[1]=\sqrt{(1,1)}=\sqrt{\Sigma x'^2_r}$.

$$\text{Now } \triangle(w)=\frac{w_1}{[1]}=o \text{ (\textit{vide} Paper III, Arts. 1 and 2)}$$

$$\text{and } \triangle_2(w)=\begin{vmatrix} (1,1), & w_1 \\ (1,2), & w_2 \end{vmatrix} \div [1][1,2]=\frac{1}{u_1}$$

For, $w_1=\frac{1}{2}(o,o)_1=\frac{1}{2}.\ o=o$

$w_2=\frac{1}{2}\ (o,o)_2=(1,1)=1$,

$(1,2)=\frac{1}{2}\ (1,1)'=o$

and u_1 (the first curvature) $=\dfrac{[1,2]}{[1]^3}=[1,2]$.

$$\text{Therefore, } \triangle_3(w)=\frac{\triangle'_2(w)+\triangle_1(w).\ u_1}{u_2}$$

$$=-\frac{u'_1}{u^2{}_1u_2}$$

$$\triangle_4(w)=\frac{\triangle'_3(w)+\triangle_2(w).\ u_2}{u_3}$$

$$=\frac{-u_1u''_1u_2+2u'^2_1u_2+u_1u'_1u'_2+u^2_1u^3_2}{u^3{}_1u^2{}_2u_3}$$

and so on.

3. An equation of the form

$$\Sigma\{(X_s-x_s)^2+2A_s(X_s-x_s)\}=o.\ (s=1,2,\ ..\ n) \qquad (iv)$$

represents what we may call a general *n-spheric* through the point x. The n co-efficients will become determined if the n-spheric is made to pass through any additional n points which are independent. Thus an n-spheric is uniquely determined by $n+1$ independent points. A system of such n-spherics will pass through $r+1$ points where r is less than n. The common intersection of the system of n-spherics will be an *r-spheric* through the $r+1$ given points, having its centre in the r-space determined by the $r+1$ given points.

If x be a point on a curve in an n-space, the osculating n-spheric at x will pass through x and an additional n points on the curve, consecutive to x. Similarly the osculating r-spheric at x will pass through x and an additional r points $(r \angle n)$ consecutive to x. The co-efficients A_s are determined for the osculating n-spheric by differentiating (*iv*) with respect to t, n times successively, considering X variable and x constant, and *then* putting $X=x$ in the result of each differentiation. We thus obtain

$$\left.\begin{array}{l} w_1+\Sigma A_s x'_s=0 \\ w_2+\Sigma A_s x''_s=0 \\ \dotfill \\ w_n+\Sigma A_s x_s^{(n)}=0 \end{array}\right\} \qquad (v)$$

The result of eliminating the n co-efficients A_s between (*iv*) and (*v*) is

$$\begin{vmatrix} \Sigma(X_s-x_s)^2, & 2(X_1-x_1), & 2(X_2-x_2), & \cdot & \cdot & 2(X_n-x_n) \\ w_1 & , x'_1 & , x'_2 & \cdot & \cdot & , x'_n \\ w_2 & , x''_1 & , x''_2 & \cdot & \cdot & , x''_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ w_n & , x_1^{(n)} & , x_2^{(n)} & \cdot & \cdot & , x_n^{(n)} \end{vmatrix}=0 \qquad (vi)$$

which is the equation of the osculating n-spheric to the given curve at x.

If equation (*vi*) be referred to the n principal axes at x it takes the form

$$\begin{vmatrix} \Sigma\xi_s^2 & , 2\xi_1 & , 2\xi_2 & , & \cdot & , 2\xi_n \\ w_1 & , \Delta_1(1) & , \Delta_2(2) & , & \cdot & , \Delta_n(1) \\ w_2 & , \Delta_1(2) & , \Delta_2(2) & , & \cdot & , \Delta_n(2) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ w_n & , \Delta_1(n) & , \Delta_2(n) & , & \cdot & , \Delta_n(n) \end{vmatrix}=0 \qquad (vii)$$

a result which may be obtained by multiplying (*vi*) by

$$\begin{vmatrix} 1 & , & 0 & , & 0 & , & . & , & 0 \\ 0 & , & \Delta_1(x_1) & , & \Delta_1(x_2) & , & . & , & \Delta_1(x_n) \\ 0 & , & \Delta_2(x_1) & , & \Delta_2(x_2) & , & . & , & \Delta_2(x_n) \\ . & , & . & , & . & , & . & , & . \\ 0 & , & \Delta_n(x_1) & , & \Delta_n(x_2) & , & . & . & \Delta_n(x_n) \end{vmatrix}$$

vide equation a Paper III.

A Δ-transformation of the last n rows of (*vii*) gives

$$\begin{vmatrix} \sum \xi_i^2 & , & 2\xi_1 & , & 2\xi_2 & , & . & , & 2\xi_n \\ \Delta_1(w) & , & 1 & , & 0 & , & . & , & 0 \\ \Delta_2(w) & , & 0 & , & 1 & , & . & , & 0 \\ . & , & . & , & . & , & . & , & . \\ \Delta_n(w) & , & 0 & , & 0 & , & . & , & 1 \end{vmatrix}$$

or $\sum \xi_i^2 - 2\xi_1 \Delta_1(w) - 2\xi_2 \Delta_2(w) - \ldots - 2\xi_n \Delta_n(w) = 0$ (*viii*)

Hence the co-ordinates of the centre of the osculating n-spheric, referred to the n principal axes at x, are

$$\Delta_1(w), \Delta_2(w), \ldots, \Delta_r(w)$$

and the radius R_n is given by

$$R_n^2 = \triangle_1^2(w) + \triangle_2^2(w + \ldots + \triangle_n^2(w). \qquad (ix)$$

a solution which in point of completeness and elegance leaves apparently nothing to be desired. We have already shewn how to express with facility $\Delta_1(w)$, $\Delta_2(w)$ &c. in terms of the principal curvatures u_1, u_2, &c. and their derivatives.

4. The equations of the *osculating r-spheric* are obtained in the same way. They are

$$\begin{Vmatrix} \sum_1^n (X_s - x_s)^2 & , & 2(X_1 - x_1) & , & 2(X_2 - x_2) & . & . & . & 2(X_n - x_n) \\ w_1 & , & x'_1 & , & x'_2 & . & . & , & x'_n \\ w_2 & , & x'_1 & , & x' & . & . & , & x''_n \\ . & , & . & , & . & . & . & , & \\ w_r & , & x_1^{(r)} & , & x_2^{(r)} & . & . & , & x_n^{(r)} \end{Vmatrix} = 0 \qquad (x)$$

where r is less than n.

These equations include the equations of the osculating r-space at x,

$$\left\| \begin{array}{ccccc} X_1-x_1 & , X_2-x_2 & , & \cdot\ \cdot\ , & X_n-x_n \\ x'_1 & , x'_2 & , & \cdot\ \cdot\ , & x'_n \\ x''_1 & , x''_2 & , & \cdot\ \cdot\ , & x''_n \\ \cdot & \cdot & & \cdot\ \cdot\ \cdot & \cdot \\ x_1^{(r)} & , x_2^{(r)} & , & \cdot\ \cdot\ , & x_n^{(r)} \end{array} \right\| =0 \qquad (xi)$$

of which the equations referred to the principal axes at x are

$$\xi_{r+1}=0\ ,\ \xi_{r+2}=0\ ,\ .\ .\ .\ \xi_n=0\ ,$$

for, the osculating r-space is the join of the first r principal axes at x.

If we multiply (x) by

$$\left\| \begin{array}{ccccc} 1 & , 0 & , 0 & , \cdot\ \cdot\ , & 0 \\ 0 & , \Delta_1(x_1) & , \Delta_2(x_2) & , \cdot\ \cdot\ , & \Delta_1(x_n) \\ 0 & , \Delta_2(x_1) & , \Delta_2(x_2) & , \cdot\ \cdot\ , & \Delta_2(x_n) \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ 0 & , \Delta_r(x_1) & , \Delta_r(x_2) & , \cdot\ \cdot\ , & \Delta_r(x_n) \end{array} \right\| \qquad (xii)$$

that is, if we multiply each determinant in (x) by the corresponding determinant in (xii) and take the sum of all the products, we have

$$\left| \begin{array}{ccccc} \sum_1^r \xi_s^2 & , 2\xi & , 2\xi_2 & , \cdot\ \cdot\ , & 2\xi_r \\ w_1 & , \triangle_1(1) & , \triangle_1(2) & , \cdot\ \cdot\ , & \triangle_2(r) \\ w_2 & , \triangle_2(1) & , \triangle_2(2) & , \cdot\ \cdot\ , & \triangle_2(r) \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ w_r & , \triangle_r(1) & , \triangle_r(2) & , \cdot\ \cdot\ , & \triangle_r(r) \end{array} \right| =0 \ (xiii)$$

where $\sum_1^r \xi_s^2$ has been written instead of $\sum_1^n \xi_s^2$, as $\xi_s=0$ for $s=r+1\ ,\ .\ .\ ,\ n\ ,$

From (*xiii*) we obtain by $\triangle$ – transformation of the last r rows,

$$\begin{vmatrix} \sum_1^r \xi_s^2 , & 2\xi_1 , & 2\xi_2 , & . & . & , & 2\xi_r \\ \triangle_1(w) , & 1 , & 0 , & . & . & , & 0 \\ \triangle_2(w) , & 0 , & 1 , & . & . & , & 0 \\ . & . & . & . & . & . & . \\ \triangle_r(w) , & 0 , & 0 , & . & . & , & 1 \end{vmatrix} = 0$$

$$\sum_1^r \xi_s^2 - 2 \sum_1^r \triangle_s(w)\, \xi_s = 0 \qquad (xiv)$$

which together with $\xi_{r+1} = 0 \;, \; . \; . \;, \; \xi = 0$

constitute the equations of the osculating r-spheric at x.

We have therefore the co-ordinates of the centre of the osculating r-spheric, referred to principal axes at x, given by

$$\triangle_1(w) \;,\; \triangle_2(w) \;,\; . \; . \;,\; \triangle_r(w) \;,\; 0 \;,\; . \; . \;,\; 0$$

and the radius R_r given by

$$R_r^2 = \triangle_1^2(w) + \triangle_2^2(w) + \ldots + \triangle_r^2(w).$$

We have evidently

$$R_r^2 - R_{r-1}^2 = \triangle_r^2(w).$$

Also because $\triangle_r(w) \,.\, u_{r-1} = \triangle'_{r-1}(w) + \triangle_{r-2}(w) \,.\, u_{r-2}$

$$\text{or, } \triangle_r(w) \,.\, \triangle_{r-1}(w) \,.\, u_{r-1} - \triangle_{r-1}(w)\, \triangle_{r-2}(w) \,.\, u_{r-2}$$
$$= \triangle_{r-1}(w) \,.\, \triangle'_{r-1}(w)$$

We have, by putting successively r equal to 3, 4, . . , r in the above equation and adding,

$$\triangle_r(w) \,.\, \triangle_{r-1}(w) \,.\, u_{r-1} - \triangle_2(w) \,.\, \triangle_1(w) \,.\, u_1$$
$$= \triangle_2(w) \,.\, \triangle'_2(w) + \triangle_3(w) \,.\, \triangle'_3(w) + \ldots + \triangle_{r-1}(w)\, \triangle'_{r-1}(w)$$

or, since $\triangle_1(w)=0$, we have

$$\triangle_r(w) \,.\, \triangle_{r-1}(w) \,.\, u_{r-1} = \triangle_1(w)\, \triangle'_1(w) + \triangle_2(w) \,.\, \triangle'_2(w) + \,.\, + \triangle_{r-1}(w)\, \triangle'_{r-1}(w).$$

But $R^2_{r-1} = \triangle^2_1(w) + \triangle^2_2(w) + \,.\, + \triangle^2_{r-1}(w)$

Therefore

$$R_{r-1}\, R'_{r-1} = \triangle_r(w)\, \triangle_{r-1}(w) \,.\, u_{r-1} \quad . \qquad (xv)$$

which measures the rate of variation of R_{r-1} .

CIRCLES INCIDENT ON AN OVAL OF UNDEFINED CURVATURE

By

S. Mukhopadhyaya (1930)

1. *Positive sense on an oval.*

The elementary convex oval discussed in this paper will be supposed to be a continuous curve. It is described by a point P moving always in the same sense, which will be called the *positive sense*, along the curve.

A definite tangent will be supposed to exist at each point P of the oval V. The positive sense along the tangent agrees with the positive sense along the oval. The oval lies on the positive side (right side) of each tangent. The tangent turns *continuously* on the positive side as P moves along V in the positive sense.

Looked at from a point inside the oval, the positive sense along the oval is clock-wise. The same holds for a circle or any other oval in the same plane.

The oval will not be postulated to possess a *definite* circle of curvature at any point.

2. *Circles incident on the oval.*

A circle which passes through at least three definite points of V will be called a circle *incident* on V.

A circle which passes through any three distinct points of V, is a circle incident on V.

A circle which touches V at a point P and passes through another point Q, distinct from P, is a circle incident on V, as the oval possesses a definite tangent at each point.

A circle which touches as well as crosses V at P is a circle incident on V, as the circle will meet V in at least one other definite point, distinct from P.

A circle which passes through only two definite points of V will not be considered incident on V. Thus a circle which touches V internally or externally, will not be considered incident on V, unless it meets the oval again at a different point.

3. *Incidence, contact and osculation, positive and negative.*

If a circle C positively described, cross V at P, passing from outside to inside (or from inside to outside), but do not touch V at P, the circle C will be said to have an *incidence* +P (or −P) on V.

If a circle C positively described, touch V externally (or internally), the circle C will be said to have incidences $\pm$P (or $\mp$P) on V.

Incidences $\pm$P (or $\mp$P) are called *contact* +P (or −P) on V.

If a circle C positively described, touch as well as cross V at P, from outside to inside (or from inside to outside), the circle C will be said to have incidences $\overset{\pm}{+}$P (or $\overset{-}{\pm}$P) on V.

In the first case the circle C is also said to have contacts $\pm$P on V and in the second case contacts $\mp$P on V. Contacts $\pm$P (or $\mp$P) may again be called *osculation* +P (or −P).

If a circle C touch V at P externally (or internally), passing through four consecutive points at P, it will be said to have an *osculation* $\pm$P (or $\mp$P) on V. Such circles will not be called incident on V, as of the four consecutive points only two count as definite, unless C has incidence on V again at some other point.

Incidences P(1), P(2)....., P(r) of a circle C on V with alternately contrary signs, will be called *successive* if they are in positive order on V and other incidences besides these do not exist between P(1) and P(r).

If P(1), P(2), ..., P(r), continuing to be successive, attain a common limit, they become *consecutive*.

4. *Upper and lower circles of curvatures. Possible circles of curvature.*

A circle whose radius is neither zero nor infinity will be called a *proper* circle.

At each point P of V we will suppose that there exists a class of proper circles C ($\pm$P) which touch V externally at P, and also a class of proper circles $C\ (\mp P)$ which touch V internally at P. The former class will have a proper circle $\underline{C\ (\pm P)}$ as lower bound, which we will call the *upper circle of curvature* at P and the latter class a proper circle $\overline{C\ (\mp P)}$ as upper bound, which we will call the *lower circle of curvature* at P.

We will suppose that there exists a proper circle $\overline{C}$ which is the upper bound of all upper circles of curvature and a proper circle $\underline{C}$ which is the lower bound of all the lower circles of curvature. The former will be called the *upper bound of possible circles of curvature* and the latter the *lower bound of possible circles of curvature* of V. The radius of the former will be denoted by $\overline{r}$ and that of the latter by $\underline{r}$.

At each point P there will generally exist a third class of circles which touch V at P as well as cross it, either from outside to inside or from inside to outside. In the former case the class will be called one of *positive* osculation and in the latter case one of *negative* osculation and the point P will be accordingly called a point of positive osculation or one of negative osculation. A point P cannot obviously be one of both positive and negative osculation. But it may so happen that osculating circles of neither kind is present at a given point. Such a point might be called a point of *plus-minus* or *minus-plus* osculation as the case may be.

At a point of positive osculation P, there is a class of circles which have positive osculation on V. The upper bound of this class is the same as the circle of upper curvature at P. It is either a circle of positive osculation or of plus-minus osculation. Similarly the lower bound of the class is the circle of lower curvature at P. It is either a circle of positive osculation or of minus-plus osculation.

At a point of negative osculation P, there is a class of circles which have negative osculation on V. The upper bound of this class, which is the upper curvature at P, is either a circle of negative osculation or of plus-minus osculation. The lower bound of the class, which is the same as the lower curvature at P, is either a circle of negative osculation or of minus-plus osculation.

Any circle of positive, negative, plus-minus or minus-plus osculation is a *possible* circle of curvature of V.

It is of interest to note that the upper curvature and the lower curvature at a given point P may be simultaneously a circle of plus-minus osculation and minus-plus osculation. Such a point may be called *bi-cyclic*.

If we define a possible circle of curvature as a circle which touches and crosses V at the same point or *is the limit of a sequence of such circles at the same or different points*, the existences of $\underline{C}$ and $\bar{C}$ as possible circles of curvature easily follows.

A circle which touches as well as crosses V at P will be called a circle of *proper* osculation, which may be either of *positive* or of *negative* sign.

5. *Deflexion. Semi-oval.*

If P moves from a position L to a position M on V in the positive sense, it describes an arc which may be denoted by V(LM). The angle through which the tangent to V at P rotates as P passes from L to M, will be called the *deflexion* of V(LM), and denoted by d.V(LM).

The angle supplementary to any angle PQR will be similarly denoted by d.PQR.

If d.V(LM) be equal to two right-angles, the arc V(LM) will be called a *semi-oval*. If *d.V*(LM) be less (or greater) than two right-angles, the arc V(LM) will be called less (or greater) than a semi-oval. A semi-oval is only a notation of convenience and does not imply necessarily a half-length of the oval.

THEOREM I.

If a circle C of radius r has successive incidences (−L, +M, −N) *on V, where V*(LN) *does not exceed a semi-oval, then there exists an equal circle 'C, which has successive incidences* (∓L′, −N) *on V*(LN), *where C′* (L′N) *is less than a semi-circle and* L′ *falls between* L *and* M.

The incidences (−L, +M) of C on V are supposed non-consecutive, as otherwise the theorem does not arise but the incidences (+M, −N) may be consecutive.

The arc C(LN) may or may not exceed a semi-circle. In the former case another equal circle C'' may be shewn to exist which has successive incidences (−L, +M″, −N) on V(LN), such that C''(LN) is less than a semicircle, having length conjugate to that of C(LN). See Fig. 1.

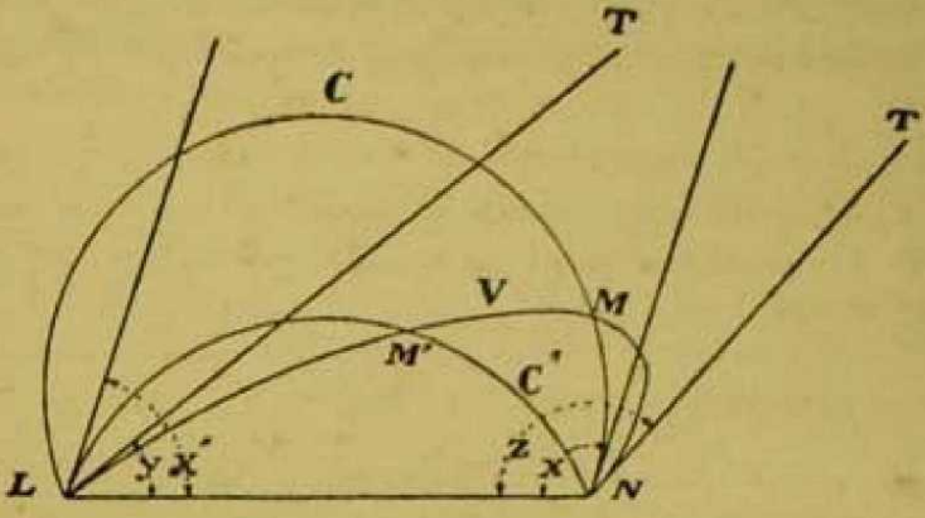


FIG. 1.

Suppose X and X'' are the base angles of C(LN) and C''(LN), respectively, where X and X'' are supplementary, the former being obtuse and the latter acute.

The tangents to V at L and N are either parallel or meet at T, where T lies on the same side of LN as M. Denote the angles TLN and TNL by Y and Z, respectively.

Then by hypothesis $Z \geqq X > X''$. Therefore the incidence of C'' at N is *negative*.

Again, $Y + Z \leqq X + X''$. Therefore, $Y \leqq X'$.

Hence the incidence of C'' at L is either of negative or ambiguous sign.

In the former case, there exists on V(LN), between L and N, at least one positive incidence of C''. If there exist more than one incidence of C'' on V between L and N, take the last two incidences.

In the latter case there may or may not exist any incidences of C'' on V between L and N. Under the former contingency a positive incidence of C'' on V nearest to N exists, between L and N, and under the latter contingency C'' has incidences $\mp$L, and the theorem does not arise.

Thus the case in which C(LN) exceeds a semi-circle reduces to the case in which C(LN) is less than a semi-circle. We proceed to consider this latter case.

Consider the class of circular arcs G of given radius r, each individual G(X) of which passes through an interior point of C(LN) and has its upper extremity at N and lower extremity at some point X on LN between L and N. See Fig. 2.

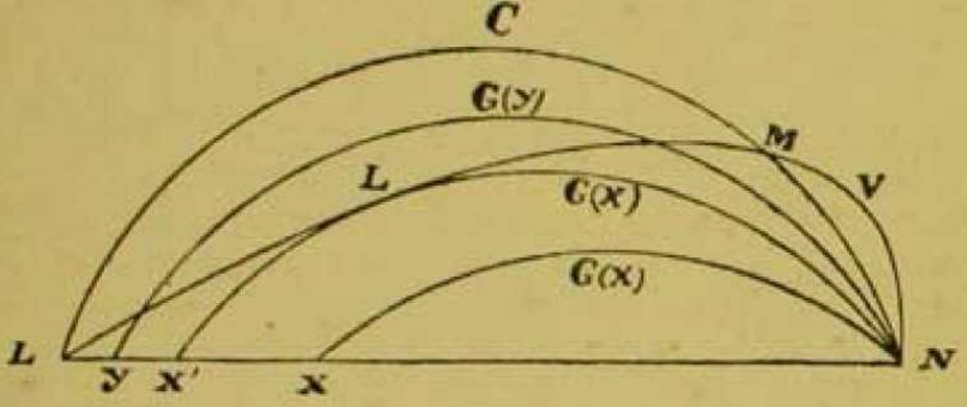


Fig. 2.

Each individual G(X) is less than a semi-circle, and is uniquely determined by the point X.

An individual G(X) will be considered of lower rank than another G(Y) if X is nearer to N than Y.

The class G is divisible Dedekind-wise into two parts G(1) and G(2). The part G(1) consists of those individuals of G which meet V between L and M and the part G(2) consists of those individuals of G which do not meet V(LM).

Each individual of G(1) is of higher rank than each individual of G(2). If an individual of G belong to G(2), every individual of lower rank will belong to G(2), and if an individual of G belong to G(1) every individual of higher rank will belong to G(1). An infinity of individuals belong to each of the classes G(1) and G(2).

A unique cut X′ of LN, lying between L and N, therefore exists, such that G(X′) is either the lowest of G(1) or the highest of G(2). Obviously G(2) has no highest individual, as if an individual belong to G(2) there exist always an infinity of individuals of higher rank which belong to G(2). Therefore G(X′) is the lowest individual of G(1).

This individual G(X′) obviously touches V(LM) internally at one or more points L′, lying between L and M, and lies entirely within V(LN). If there is more than one point L′, choose the one nearest to N.

Cor. If a circle C of radius r has successive incidences (+L, −M, +N) *on V, where V*(LN) *does not exceed a semi-oval, then there exists an equal circle C' which has successive incidences* (+L, ∓N′) *on V*(LN), *where C'*(LN′) *is less than a semi-circle and* N′ *falls between* M *and* N.

THEOREM II.

If a circle C of radius r has successive incidences (−L, +M, −N) *on an oval V, where V*(LN) *does not exceed a semi-oval, then there exists an equal circle C^* which has successive incidences* ($-$L*, $\pm$N*) *on V where* L*, N* *lie on V*(LN), *between* L *and* N *and* C^*(L*N*) *is less then a semi-circle.*

By Theorem I, an equal circle C' exists which is incident on V(LN) at successive points (∓L′, −N), such that C(L′N) is less than a semi-circle and L′ lies between L and M.

Suppose H is the highest point of V(L′N) and XY a chord of V(L′N) parallel to L′N, where X lies on V between L′ and H.

Then because V(L′N) is less than a semi-oval, XY is less than L′N. Produce XY to Z making XZ equal to L′N. Describe on XZ a circular arc G(X) congruent to C'(L′N) on the side of XZ on which H lies. See Fig. 3.

Then because XY is inclined to V at X at an angle smaller than that at which L′N is inclined at L′, G(X) will have incidence −X on V. As G(X) goes outside V at X and is also outside V at Z, it will either meet V at an even number of points, between X and Y, or not meet V at all.

The class G of circular arcs G(X), therefore, fall into two divisions, G(1) and G (2), the former consisting of those individuals of G which meet V and the latter of those which do not meet V, between X and Z.

Each individual G(X) of G is uniquely determined by the point X, whose position varies continuously on V between L′ and H. G(X) has a rank which ascends as X moves in the positive sense from L′ to H.

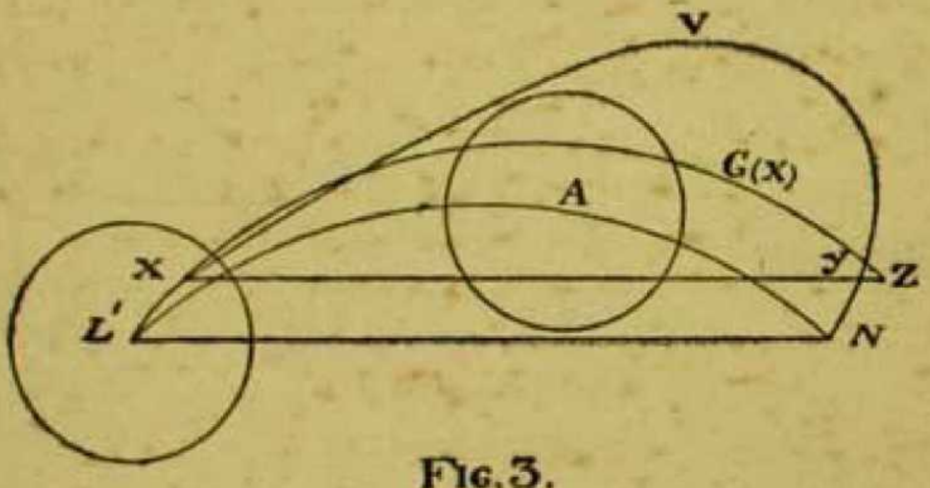
We shall shew that

(*i*) G(1) and G(2) both exist and are infinitely numerous.

(*ii*) Each individual of G(2) is of higher rank than each individual of G(1).

(*iii*) G(1) and G(2) determine a Dede-kind section L^* of V(L′H), such that G(L^*) belongs to G(1).

To prove that G(1) exists, take any point A on C' (L′N) between L′ and N and with A as centre describe a circle c of sufficiently small radius u such that c falls entirely within V(L′N). Draw an equal circle c' wilh L′ as centre. Now if X fall within c', G(X) will obviously pass through an interior point of c and must, consequently, in passing from X to Z meet V(L′N) at an even number of points. An infinity of such points X on V(LH) can be selected as lie within c'. See Fig. 3.



Fig, 3.

To prove that G(2) exixts, one has to notice that d.V(XY) continuously diminishes to limit zero as X moves from L′ to H, and that d.G(X) is a constant which may be denoted by k.

Choose a position B for X such that d.V(XY) is less than $\frac{1}{2}k$. Then for any position of X between B and H, G(X) cannot meet V again. For the necessary and sufficient condition that a point P of V(XY) will fall within G(X) is that d.(XPZ) is less than $\frac{1}{2}$k. But d.(XPZ) $<$ d.(XPY) $<$ d.V.(XY) $<$ $\frac{1}{2}$k, by hypothesis.

It is also seen that if G(X) belong to G(2) for a given position of X, it will belong to G(2) for all higher positions of X.

For if X′ be higher than X and Y′, Z′ correspond to Y, Z and if P be any point on V(X′Y′), then d.(X′PZ′) is obviously less than d.(XPZ) and consequently less than $\frac{1}{2}k$.

Hence an individual of G(2) cannot have an individual of G(1) above it, that is, every individual of G(1) lies below every individual of G(2).

Consequently a Dede-kind section L^* of $V(L'H)$ lying between L' and H, exists, such that either $G(L^*)$ is the lowest of $G(2)$ or highest of $G(1)$.

But obviously a lowest of $G(2)$ does not exist, for if $G(X)$ belong to $G(2)$, a lower individual $G(X')$ can be found which also belongs to $G(2)$. This follows from the principle of continuity and the fact that if, for a point P on $V(XY)$, d.(XPZ) is less than $\frac{1}{2}k$, then an X' lower than X exists, sufficiently near to X, such that d.($X'PZ'$) continues less than $\frac{1}{2}k$, so that, above the line XY, $G(X')$ falls outside $V(XY)$, and between the lines $X'Y'$ and XY, $G(X')$ is also outside V as $G(X')$ has a negative incidence on V at X'.

Hence $G(L^*)$ belongs to $G(1)$ and is the highest of $G(1)$. $G(L^*)$ will touch $V(L,N)$ externally at one or more points N^*, between L' and N but otherwise lie entirely outside V. If not, an individual of $G(1)$ higher than $G(L^*)$ would exist. If there are more than one point N^*, the one nearest to L^* should be taken.

Thus Theorem II is completely proved.

THEOREM III.

If a circle C have successive incidences $(-L, \pm M)$ on V, where $C(LM)$ does not exceed a semi-circle, then an equal circle C' exists which has successive incidences $(\pm L', -M)$ on V, where L' lies on V between L and M, and a third equal circle C'' exists which has successive incidences $(-L', \pm M')$ on V where M' lies on V between L' and M, and $C(LM) \geq C'(L'M) \geq C''(L'M')$.

To prove the first part of the theorem, take any point X on LM, between L and M. Let $G(X)$ denote the minor arc of an equal circle standing on XM. Evidently $G(X)$ falls within $C(LM)$, is less than $C(LM)$, and has incidence $-M$ on V.

Consider the class G of individuals $G(X)$ the rank of each individual being estimated by the distance LX. The individuals of class G fall into two divisions $G(1)$ and $G(2)$, $G(1)$ consisting of those individuals which do not meet V between X and M, and $G(2)$ of those individuals which meet V between X and M necessarily at an even number of points.

By the method of last theorem a Dede-kind section X' of LM, between L and M, exists, such that $G(X')$ forming part of a circle C' equal to C, touches V internally at one or more points L', between L and M, but otherwise lies entirely within V with the exception of the extremity M which lies on V. Evidently $C(\mathrm{LM}) \geq \mathrm{G}(\mathrm{X}') > C'(\mathrm{L}'\mathrm{M})$.

To prove the second part, produce L′M to N, making L′N equal to the diameter of C, and on L′N describe the semi-circle $C^*(\mathrm{L}'\mathrm{N})$. Produce the tangent to V at L′, positively, to meet C at P. See Fig. 4.

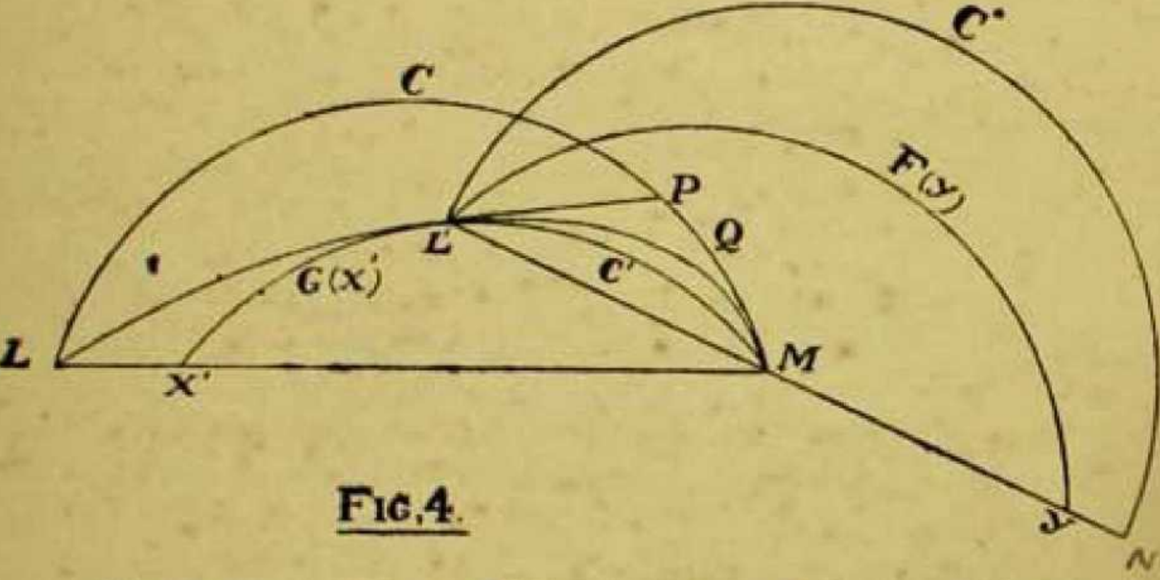


FIG. 4.

Take any point Q on V(PM), including P but excluding M. The angle L′QM which is greater than the angle LQM, is obtuse. Much more is the angle L′QN obtuse. Hence Q falls within $C^*(\mathrm{L}'\mathrm{N})$. Much more does $V(\mathrm{L}'\mathrm{M})$ fall within $C^*(\mathrm{L}'\mathrm{N})$.

The circle $C'(\mathrm{L}'\mathrm{M})$ touches V internally at L′ and crosses out V at M. The angle PL′M is acute, as $C'(\mathrm{L}'\mathrm{M})$ is less than a semi-circle. $V(\mathrm{L}'\mathrm{M})$ falls between $C^*(\mathrm{L}'\mathrm{N})$ and $C'(\mathrm{L}'\mathrm{M})$.

Take any point Y on MN between M and N, and describe a minor circular arc of radius equal to that of C on L′Y. This arc will fall between $C^*(\mathrm{L}'\mathrm{N})$ and $C'(\mathrm{L}'\mathrm{M})$. Denote it by F(Y).

Consider the class F of circular arcs F(Y). Each individual of F has incidence $-\mathrm{L}'$ on V. Each individual F(Y) will have a rank estimated by the distance MY. The class F will have two divisions F(1) and F(2), F(2) consisting of those which do not meet V between L′ and M and F(1) of those which meet V between L′ and M, necessarily at an even number of points. Each individual of F(2) will have higher rank than each individual of F(1).

From considerations, explained in the last theorem a Dede-kind section Y* of MN exists between M and N, such that F(Y*) touches

V externally at one or more points M′, between L′ and M, but otherwise lies entirely outside V, except at the extremity L′. If there is more than one point M′, we take the one nearest to L′. If F(Y*) belong to a circle C'', then C'' has successive incidences $(-\mathrm{L}', \pm\mathrm{M}')$ on V.

Evidently $C'(\mathrm{L'M}) > C''(\mathrm{L'M'})$. For as M′ falls within C(LM), the angle LM′M is obtuse and the angle L′M′M is much more obtuse. Hence L′M′ is smaller than L′M and therefore $C''(\mathrm{L'M'})$ is smaller than $C'(\mathrm{L'M})$.

Cor. No circle incident on an oval can be an absolute maximum or an absolute minimum.

For if C be a circle of radius r incident on V, then three successive incidences $(-\mathrm{L}, +\mathrm{M}, -\mathrm{N})$ always exist, with at most two consecutive, such that V(LN) does not exceed a semi-oval.

Therefore from Theorem II an equal circle C' exists which has successive incidences $(-\mathrm{L}', \pm\mathrm{M}')$ on V, such that C′(L′M′) is less than a semi-circle, where L, L′, M′, N are in order.

Again from first part of Theorem III, another equal circle C'' exists which has successive incidences $(\mp\mathrm{L}'', -\mathrm{M}')$ on V, where L″ lies between L′ and M′. Take a point P on V between L″ and M′. Then obviously the circle L′PM′ has a radius greater than r and the circle L″PM′ has a radius smaller than r, and these are incident circles.

THEOREM IV.

(*a*). *If a circle C of radius r has successive incidences $(-\mathrm{L}, +\mathrm{M}, -\mathrm{N})$ on an oval V and if V(LN) do not exceed a semi-oval, then there exists on V(LN), between L and N, a point O at which a possible circle of negative osculation exists of radius r.*

(*b*). *If a circle C of radius r has successive incidences $(+\mathrm{L}, -\mathrm{M}, +\mathrm{N})$ on an oval V and if V(LM) do not exceed a semi-oval, then there exists on V(LN), between L and N, a point O, at which a possible circle of positive osculation exists of radius r.*

To prove (*a*):

By Theorem II, there exists a circle C^* of radius r which has successive incidences $(-\mathrm{L}^*, \pm\mathrm{M}^*)$ on V(LN), where L, L*, M*, N are in order and $C^*(\mathrm{L^*M^*})$ is less than a semi-circle.

By Theorem III, there exist circles C' and C'' of radius r which have successive incidences $(\mp\mathrm{L}', -\mathrm{M}^*)$ and $(\mathrm{L}', \pm\mathrm{M}')$, respectively, on $V(L^*M^*)$ where L^*, L', M', M^* are in order and

$$C^*(L^*M^*) > C'(L'M^*) > C''(L'M').$$

By repeated applications of Theorem III, we arrive at two sequences of points $L^*, L', L'', \ldots, L^{(2n)}, \ldots,$ and $M^*, M', M'', \ldots, M^{(2n)}, \ldots,$ the first in positive order and the second in negative order, on V, such that each point of the first sequence is below each point of the second sequence, and two diminishing sequences of circular arcs $C^*(L^*M^*), C''(L'M'), \ldots, C^{(2n)}(L^{(n)}, M^{(n)}) \ldots$ and $C'(L'M^*), C'''(L''M'), \ldots, C^{(2n+1)}(L^{(n+1)}, M^n), \ldots,$ the former having incidences $(-L^*, \pm M^*), (-L', \pm M'), \ldots, (-L^{(n)} \pm M^{(n)}), \ldots,$ and the latter incidences $(\mp L', -M^*), (\mp L'', -M'), \ldots, (\mp L^{(n+1)}, -M^{(n)}), \ldots,$ respectively, on V.

The two sequences of points will have a common limit O. If not, suppose the first sequence has a limit O' and the second sequence a different limit O'', O'' being necessarily above O'.

The sequence of equal circles $C^*, C'', \ldots, C^{(2n)}, \ldots$ having successive incidences $(-L^*, \pm M^*), (-L', \pm M'), \ldots, (-L^{(n)}, M^{(n)}), \ldots,$ respectively, on V, will tend to a limiting circle $C^{(2\omega)}$, as n tends to ω (infinity), which will have successive incidences $(-O', \pm O'')$ on V and each circle of the sequence and consequently $C^{(2\omega)}$ will fall *above* $V(O'O'')$, between O' and O''.

Similarly, the sequence of equal circles $C', C''', \ldots, C^{(2n+1)}, \ldots,$ having incidences $(\mp L', -M^*), (\mp L', -M'), \ldots, (\mp L^{(n+1)}, -M^{(n)}), \ldots,$ respectively, on V, will tend, to a limiting circle $C^{(2\omega+1)}$, as n tends to ω (infinity), which will have successive incidences $(\mp O', -O'')$ on V and fall *below* $V(O'O'')$, between O' and O''.

Hence two circles $C^{(3\omega)}$ and $C^{(2\omega+1)}$ of equal radius r will pass through the same points O' and O'' and have their minor arcs different and on the same side of $O'O''$, which is impossible.

Hence O' and O'' must coincide, at a single point O. Thus $C^{(2\omega)}$ and $C^{(2\omega+1)}$ will become one circle having incidences $(-, +, -)$ O on V, so that it is a possible circle of osculation at O, of radius r and negative sign.

Theorem IV(*b*) is proved exactly in the same way.

Cor. If a circle C of radius r has osculation of a given sign at a point O on V, then there exists at least another circle C′ of the same radius which has osculation of contrary sign at a point O′ on V, different from O.

The circle C will meet V again in at least another point P, preceding O or succeeding O, such that $V(PO)$ or $V(OP)$ does not exceed a semi-oval. Hence there exists a point O between O and P or between P and O, where an equal circle C' has osculation of contrary sign on V.

THEOREM V.

If a circle C of radius r be incident on an oval V, then there exists at least two possible circles of osculation of contrary signs which have the same radius r.

Suppose first that C has only four incidences on V, namely, $-K, +L, -M, +N$, which are all distinct.

Then of the two arcs $V(KM)$ and $V(MK)$ at least one will not exceed a semi-oval. Suppose $V(KM)$ does not exceed a semi-oval. Then there exists a point O on $V(KM)$ between K and M, with a possible circle of osculation of radius r and negative sign by Theorem IV(*a*).

Similarly of the two arcs $V(LN)$ and $V(NL)$ at least one will not exceed a semi-oval. Suppose $V(LN)$ does not exceed a semi-oval. Then there exists on $V(LN)$ between L and N, a point O with a possible circle of osculation of radius r and positive sign by Theorem VI (*b*).

The same proof holds if one or both pairs of points of incidence are consecutive, or if three successive points of incidence are consecutive. If all the four points are consecutive, the circle C ceases to be an incident circle.

If V has more than four incidences and if we take any five successive incidences, H, K, L, M, N, beginning either with a positive or with a negative incidence, at least one of the two arcs $V(HL)$ or $V(LN)$ will be less than a semi-oval, whence the theorem follows.

Cor. i. If $\underline{r}$ and $\bar{r}$ denote the lower and upper bounds of possible radii of curvature of V, and if r be the radius of a circle of proper osculation on V then $\underline{r} < r < \bar{r}$.

The circle of radius r having proper osculation on V is a circle incident on V. Consequently, by Theorem III, Cor. circles of radii r' and r'' exist, incident on V, such that $r' < r < r''$. But r' and r'' are possible radii of curvature by Theorem V.

Hence $\underline{r} < r' < r < r'' < \bar{r}$.

Cor. ii. A circle of radius $r \leq \underline{r}$ *will roll freely inside* V, *and* V *will roll freely inside a circle of radius* $r \geqq \overline{r}$.[1]

In the first case, the free rolling will cease to exist only if the circle in any position becomes incident on V and therefore, by Theorem V, an equal circle has proper osculation on V. But by the above corollary the radius of a circle of proper osculation on V lies between $\underline{r}$ and $\overline{r}$. Hence the circle of radius $r \leqq \underline{r}$ which has in contact with V at each point and incidence on V nowhere, will roll freely inside V.

Similarly the oval V will roll freely within the circle of radius $r \geqq \overline{r}$.

Cor. iii. There exists at least two circles of proper osculation on V, *one positive and the other negative, of any given radius* r, *where* r *lies between* $\underline{r}$ *and* $\overline{r}$.

If a circle of radius r roll freely inside V, it has internal contact with V at every point P and consequently r is equal to or less than the radius of lower curvature at every point P. Hence $r \leqq \underline{r}$.

Similarly if a circle of radius r roll freely outside V (enclosing V), it has external contact with V at every point P and consequently r is equal to or less than the radius of upper curvature at every point P. Hence $r \geqq \overline{r}$.

Therefore a circle C of radius r, where r lies between $\underline{r}$ and $\overline{r}$, cannot roll freely inside or outside V (enclosing it). Hence C will become eventually incident on V, if an attempt is made to roll it inside or outside V. Thus the corollary follows from the Theorem V.

Cor. iv. If the lower bound of possible radii of curvature of an oval V *be equal to or greater than the upper bound of possible radii of curvature of another oval* V', *then* V' *will roll freely within* V.

If V' do not roll freely within V, then the two ovals will have a common incident circle and therefore a common radius of proper osculation. Conversely, if they have a common radius of proper osculation they may be made to touch where they have equal circles of proper osculation, of contrary signs. They will necessarily cross each other at the point of contact and therefore cannot freely roll, one inside another.

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[1] *Cf. A. Hurwitz*, "Uber die Fouriersche constenten integriebar Functionen." *Mathematische Annalen Bd.* 57 (1903), p. 444. *W. Blaschke*, "Kreis und Kugel" (1916), p. 116. *S. Ogiwara*, "On the elliptically curved ovals," Japanese Journal of Mathematics, Vol. III, 1926, p. 41.

*Cor. v. On an oval V there exist at least two points of positive osculation and two points of negative osculation of a circle equal to C, where C has the same perimeter (area) as the oval V.*2)

Let C (P) denote a circle of radius equal to that of C which touches the oval V at a point P and lies on the oval-side of the tangent at P. In no position of P can C (P) fall entirely inside or entirely outside V, as in the first case C (P) would have a less perimeter (area) than that of V, and in the second case greater perimeter (area) than that of V.

For every point P, C (P) will therefore be a circle incident on V. Hence from Theorem V there exist at least two circles, each equal to C, one having a positive osculation and the other a negative osculation on V.

Suppose M is a point of positive osculation thus obtained and N the point of V which is diametrically opposite to M, so that V(NM) and V(MN) are each a semi-oval.

The circle C(N) is a circle incident on V. It will therefore meet V again in two other points P and Q.

First suppose both P and Q lie on V(MN), P, Q, N being successive points of incidence. Then of the two triads of successive points P, Q, N, and Q, N, N, each lying in V(MN), one is pre-positive and the other pre-negative. The pre-positive triad will give rise to a positive point of osculation of a circle equal to C, lying on V(MN) between M and N. If P and Q both lie on V(NM), N, P, Q being successive, the same argument will establish the existence of a positive point of osculation of a circle equal to C, lying on V(NM) between N and M.

Next suppose P lies on V(NM) and Q on V(MN), the incidences Q, N, N, P being successive. Then of the triads Q, N, N and N, N, P, one will be pre-positive and the other pre-negative. Hence another

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2) *Cf. T. Hayashi*, Some Geometrical applications of Fourier's Series. Rendiconti, Palermo, Vol. 50 (1926), p. 100, where the case of osculating circles of perimeter equal to that of the oval has been considered.

point of positive osculation of a circle equal to C, will exist either between Q and N or between N and P.

The arguments will equally hold if P and Q coincide, or one of them coincide with N.

Thus at least two distinct points of positive osculation of a circle equal to C exist on V. Similarly it can be proved that at least two distinct points of negative osculation of a circle equal to C exist on V.

Cor. vi. On an oval V, of constant breadth, there exist at least three points of positive osculation and three points of negative osculation of a circle equal to C, where C has the same perimeter as the oval V.[1]

In an oval V, of constant breadth $2a$ the perimeter is equal to that of a circle C of radius a. Hence by preceding corollary there exist at least four points on V, two of each sign, where a circle of radius a has proper osculation on V.

It may be observed that if PP′ be a double normal of V, necessarily of length $2a$, there always exist two possible circles of osculation, one at P and the other at P′, the sum of whose radii is $2a$. If therefore the radius of a possible circle of osculation at P be a, that of a corresponding circle of osculation at P′ is also a. If the osculation at P has a given sign, that at P′ will have the contrary sign.

Let P be the point where a circle of radius a has osculation of positive sign on V. Then there exist at P′ an equal circle which has osculation of negative sign on V. The circle C' on PP′ as diameter has at each of the points P and P′ three consecutive incidences on V, the first incidence at P being positive and the last incidence at P′ negative. Between these two incidences there are four other incidences of C' on V, namely, P, P, P′, P′. Since V(PP′) is a semi-oval there will exist, by Theorem V, two points M and N, between P and P′, where a circle of radius a has negative and positive osculation, respectively. At the corresponding points M′ and N′, a circle of radius a will have positive and negative osculation, respectively. Hence there exist at least six points P, M, N, P′, M′, N′ on V, at each of which a circle of radius a has proper osculation, of positive sign at P, N, M′ and of negative sign at M, P′, N′.

[1] *Cf. W. Blaschke* "Archiv d. Math und Physik," Vol. 26 (1917), Aufgabe 540, p. 65.

THEOREM VI.

If a circle C of radius r have $2n$ successive incidences $+L', -L''$, $+L''', \ldots, -L^{(2n)}$ on an oval V, then there exist on V at least $n-1$ distinct points each having a circle of proper osculation of radius r and positive sign and an equal number of distinct points different from these each having a circle of proper osculation of radius r and negative sign.

The incident circle divides V into $2n$ successive arcs $V(L'L'')$ $V(L' L''), \ldots, V(L^{2n} L')$. If no two successive arcs together exceed a semi-oval, then there will exist in the interior of each of the arcs $V(L' L''), \ldots, V(L^{(2n-1)}L')$ a point with a circle of proper osculation of radius r and positive sign, and in the interior of each of the arcs $V(L'' L''''), \ldots, V(L^{2n} L'')$ a point with a circle of proper osculation of radius r and negative sign, by Theorem V.

If the sum of certain pairs of successive arcs exceed a semi-oval at most two such pairs can exist and these are obtained from three successive arcs. If we exclude successively these two pairs of arcs from our consideration we obtain $n-1$ distinct points of positive osculation with radius r and an equal number of points of negative osculation with the same radius. Hence the Theorem is proved. The above proof will equally hold if any pair or triad of successive points of incidence of C on V are consecutive.

If four successive points are consecutive they will be counted as two consecutive points, for the purposes of the Theorem.